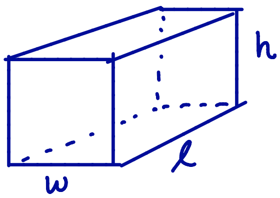
Section 4.7 Optimization (Day 2)

1. A rectangular storage container with lid is to have a volume of 36 cubic inches. The length of the base is three times the width. Material for the base and lid costs \$4 per square inch. Material for the sides costs \$1 per square inch. Find the cost of materials for the least expensive container.



$V = 36\text{ in}^3$

minimize cost

$C = \$4(w\ell)\cdot 2 + \$1(wh\cdot 2 + \ell h\cdot 2)$

$= 8w\ell + 2wh + 2\ell h$

but $\boxed{3w = \ell}$ and $36 = V = \ell w h = 3w^2 h$

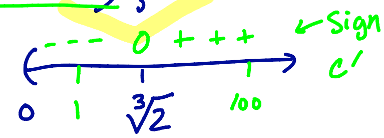
So $\boxed{h = \frac{36}{3w^2} = 12w^{-2}}$

$C(w) = 8w(3w) + 2w(12w^{-2}) + 2(3w)(12w^{-2})$

$= 24(w^2 + 4w^{-1})$ on $(0,\infty)$

$C'(w) = 24(2w - 4w^{-2})$, $C' = 0$ when $2w = \frac{4}{w^2}$ or $w = \sqrt[3]{2}$

1st. der. test:

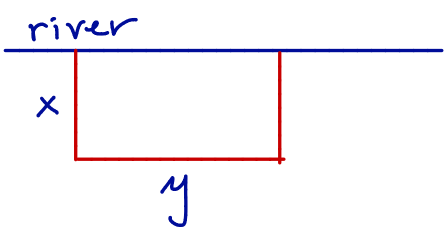


C has an absolute min when $w = \sqrt[3]{2}$. (b/c $w = \sqrt[3]{2}$ is the only crit. pts.)

ANS: Dimensions: $w = \sqrt[3]{2}$ in, $\ell = 3\sqrt[3]{2}$ in, $h = \frac{12}{2^{2/3}}$

Cost: $C(w) = 24\left(2\sqrt[3]{2} - \frac{4}{2^{2/3}}\right) = 72(2^{2/3})$ dollars

2. You have 800 feet of fencing to make a pen. Assume you have a river on your property and you will use the river for one side of the pen. What dimensions of the pen will maximize area?



$2x + y = 800$

maximize area

$A = xy$

use $y = 800 - 2x$

So $A(x) = x(800 - 2x)$

$= 800x - 2x^2$

$D: [0, 400]$

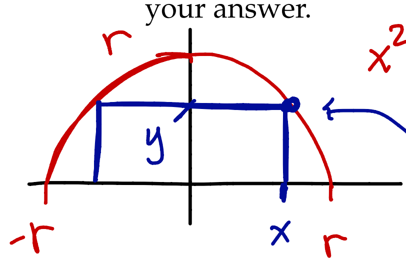
$A'(x) = 800 - 4x$

$A' = 0$ when $x = 200$ ft

$A(x)$ has an absolute maximum at $x = 200$ because $A(x)$ is a parabola that opens down

Answer: The dimensions that maximize area are $x = 200$ ft and $y = 400$ ft.

3. Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.
Hint: The radius r of your circle can be considered a fixed constant. You will expect it to appear in your answer.



$x^2+y^2=r^2$ (whole circle)

$y=\sqrt{r^2-x^2}$ (top half)

point $P(x,y)$

So $y=\sqrt{r^2-x^2}$

maximize area A

$A=2xy$

$A(x)=2x(r^2-x^2)^{1/2}$ on $[0,r]$

$A'(x)=2(r^2-x^2)^{1/2}+2x\left(\frac{1}{2}\right)(r^2-x^2)^{-1/2}(-2x)$

$=2(r^2-x^2)^{1/2}-\frac{2x}{(r^2-x^2)^{1/2}}$

$A'(x)=0$ when $2(r^2-x^2)=2x^2$

So $r^2=2x^2$ or $x=\pm r/\sqrt{2}$

$-\frac{r}{\sqrt{2}}$ outside domain

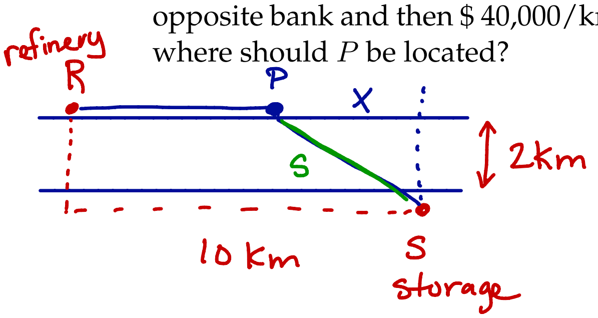
table:

x	0	r	$r/\sqrt{2}$
$A(x)$	0	0	r^2

Answer: The maximum area is

$A\left(\frac{r}{\sqrt{2}}\right)=2\left(\frac{r}{\sqrt{2}}\right)\left(\frac{r}{2}\right)^{1/2}=r^2$

4. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 10 km downstream of the refinery. The cost of laying pipe is \$ 10,000/ km over land to a point P on the opposite bank and then \$ 40,000/km under the river to the tanks. To minimize the cost of pipeline, where should P be located?



$s=\sqrt{x^2+4}$

minimize cost (in \$10,000)

$C(x)=(10-x)(1)+\left(\sqrt{x^2+4}\right)4$

$=10-x+4(x^2+4)^{1/2}$

D: $[0,10]$

$C'(x)=-1+2(x^2+4)^{-1/2}(2x)$

$=-1+\frac{4x}{\sqrt{x^2+4}}$

$C'=0$ when $4x=\sqrt{x^2+4}$ or $16x^2=x^2+4$

So $x=\pm\frac{2}{\sqrt{15}}$

table:

x	0	10	$2/\sqrt{15}$
$C(x)$	18	$8\sqrt{104}$	10.48

minimum

Answer: P should be located

$10-\frac{2}{\sqrt{15}}\approx 9.5$ km downriver.