1. L'Hôpital's Rule (again but even better)....

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\boxed{\frac{\infty}{\infty}}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} \overset{H}{=} \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided the 2nd limit exists or is $\boxed{\pm\infty}$

(Also a can be $\pm\infty$.)

2. Evaluate the following limits using any appropriate method.

(a) $\underbrace{\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x \to \infty} \frac{2\sqrt{x}}{x} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$

(b) $\underbrace{\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 2x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x \to 2} \frac{2x}{2x - 2} = \frac{4}{4-2} = 2$

(c) $\underbrace{\lim_{x \to \infty} \frac{2e^x + 1}{1 - 3e^x}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x \to \infty} \frac{2e^x}{-3e^x} = \lim_{x \to \infty} \frac{-2}{3} = \frac{-2}{3}$

(d) $\lim_{x \to 0} \frac{\cos(4x)}{3e^{3x}} = \frac{1}{3}$

Can't use L'Hopital for everything.

3. Now for some more sophisticated applications.

(a) $\lim_{x\to\infty} x\sin(\frac{\pi}{x}) = \lim_{x\to\infty} \underbrace{\frac{\sin(\pi x^{-1})}{x^{-1}}}_{\text{form } \frac{0}{0}} = \lim_{x\to\infty} \frac{\cos(\pi x^{-1})(-1\cdot\pi x^{-2})}{-x^{-2}} = \lim_{x\to\infty} \pi\cos(\frac{\pi}{x}) = \pi$

$\infty \cdot 0$

use $x = \frac{1}{\frac{1}{x}} = \frac{1}{x^{-1}}$

(b) $\lim_{x\to 0^+} (1+x)^{1/x} = \boxed{e}$

① transform problem:

old $y = (1+x)^{1/x}$

new $\ln y = \frac{1}{x}\ln(1+x) = \frac{\ln(1+x)}{x}$

② Find limit of new problem:

$\lim_{x\to 0^+} \underbrace{\frac{\ln(1+x)}{x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}\cdot 1}{1} = \lim_{x\to 0^+} \frac{1}{1+x} = \frac{1}{1} = 1$

③ Undo transformation from ①:

We found $\lim_{x\to 0^+} \ln(y) = 1$

So $\lim_{x\to 0^+} y = e^1 = e$

use ⓘ $e^{\ln(y)} = y$

ⓘⓘ if $a=b$ then $e^a = e^b$

(c) $\lim_{x\to\infty} \underbrace{\frac{e^{x/10}}{x^2}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x\to\infty} \underbrace{\frac{\frac{1}{10}e^{x/10}}{2x}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{100}e^{\frac{x}{10}}}{2} = \infty$

(d) $\lim_{x\to 1^+} \underbrace{(\ln(x^4-1) - \ln(x^9-1))}_{\text{form } \infty-\infty} = \lim_{x\to 1^+} \left(\ln\left(\frac{x^4-1}{x^9-1}\right)\right) = \ln\left[\lim_{x\to 1^+} \underbrace{\left(\frac{x^4-1}{x^9-1}\right)}_{\text{form } \frac{0}{0}}\right]$

$\overset{H}{=} \ln\left[\lim_{x\to 1^+} \frac{4x^3}{9x^8}\right] = \ln\left(\frac{4}{9}\right)$