Section 2-2: The Limit of a Function

1. DEFINITION: **two-sided limit**

   Notation: $\lim_{x \to a} f(x) = L$ a and L are numbers.

   Words: the limit of f(x) as x approaches a is L

   It means: as x-values get closer and closer to the number a, the y-values of f(x) are getting arbitrarily close to the number L.

2. Example 1: Evaluate $\lim_{x \to 2} \frac{x^2-4}{x-2}$ numerically.

   Answer: $\lim_{x \to 2} \frac{x^2-4}{x+2} = 4$

x	1	1.5	1.9	1.999	2	2.001	2.1	2.5	3
$y = \frac{x^2-4}{x-2}$	3	3.5	3.9	3.999	DNE	4.001	4.1	4.5	5

approach y=4

3. Example 2: Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$ numerically.

   answer: $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

x	-0.5	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.5
$\frac{\sin(x)}{x}$	0.95885	0.99833416...	0.9999833	0.9999998...	DNE	0.9999998...	0.9999833...	0.99833416...	0.95885...

approach 1

4. Example 3: Limits do not always exist. Evaluate each limit below numerically and explain why the limits do not exist.

   (a) $\lim_{x \to 2} \frac{|x-2|}{x-2}$

   Answer: $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist because the left-side and the right-side are not the same.

X	1	1.9	1.999	2	2.001	2.1	3
$y = \frac{\lvert x-2 \rvert}{x+2}$	-1	-1	-1	DNE	+1	+1	+1

$-1 \neq 1$

   (b) $\lim_{x \to 0} \frac{1}{x^2}$

X	-0.1	-0.01	-0.00001	0	0.00001	0.01	0.1
$y = \frac{1}{x^2}$	100	10000	10^{10}		10^{10}	10000	100

Answer: $\lim_{x \to 0} \frac{1}{x^2}$ does not exist because the y-values are unbounded. One can write $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Geometrically, we know $f(x) = \frac{1}{x^2}$ has a vertical asymptote at $x=0$.!!

It is possible to have one-sided limits. Example 4a at the bottom of page 1 illustrates this.

5. Notation:

$x \to 2^-$ means x approaches 2 on the left or from values a bit smaller than 2

and

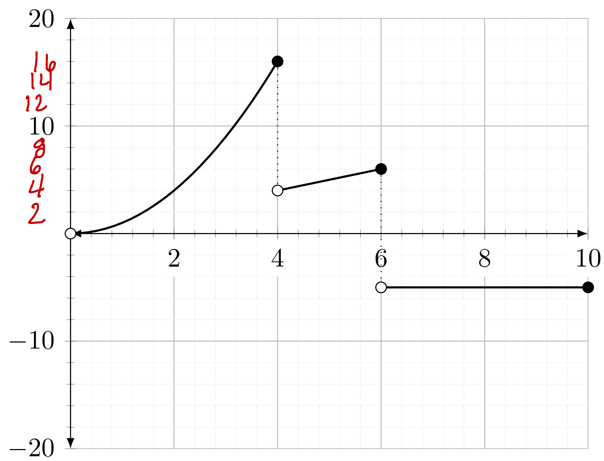
$x \to 2^+$ means x approaches 2 on the right or from values a bit larger than 2.

(a) $\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1$ (b) $\lim_{x\to 2^+} \frac{|x-2|}{x-2} = +1$

These come from the calculations in table for Example 4a

Limits can also be evaluated graphically.

6. The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} g(x) =$ 16

(b) $\lim_{x\to 4^+} g(x) =$ 4

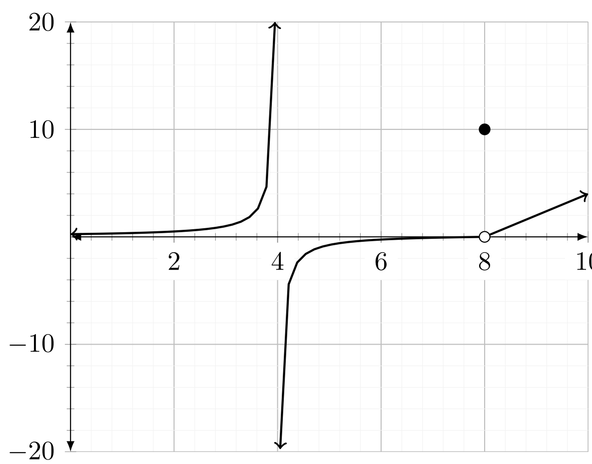
(c) $\lim_{x\to 4} g(x) =$ DNE

(d) $g(4) =$ 16

(e) $\lim_{x\to 8} g(x) =$ -5

(f) $g(8) =$ -5

7. The function $h(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} h(x) =$ $+\infty$

(b) $\lim_{x\to 4^+} h(x) =$ $-\infty$

(c) $\lim_{x\to 4} h(x) =$ DNE

(d) $h(4) =$ DNE

(e) $\lim_{x\to 8} h(x) =$ 0

(f) $h(8) =$ 10

8. Find any vertical asymptotes of $f(x) = \frac{2}{x+5}$ and *justify* your answer using a limit.

V.a.: $x = -5$ ← Value makes denominator zero.

Justification:

$$\lim_{x \to -5^+} \frac{2}{x+5} = +\infty$$

as $x \to -5^+$ (#'s like $-4.9, -4.99$)

$x+5 \to 0^+$

9. Sketch the graph of an function that satisfies *all* of the given conditions. Compare your answer with that of your neighbor.

$\lim_{x\to 0^-} f(x) = 1$ $\quad \lim_{x\to 0^+} f(x) = -2$ $\quad \lim_{x\to 4^-} f(x) = 3$ $\quad \lim_{x\to 4^+} f(x) = 0$

$f(0) = -2$ $\quad f(4) = 1$