1. Recall the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

2. Let $f(x) = e^x$. **Estimate** $f'(x)$ (a.k.a. the slope of the tangent line) using the limit definition for each of the values below. (Use a calculator!)

   (a) $f'(0)$

$$x=0:\ f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{e^h} \approx \frac{e^{0.001}-1}{e^{0.001}} = \underbrace{1.0005}_{=}$$

$$f(0) = e^0 = 1$$

   (b) $f'(1)$

$$f'(1) = \lim_{h\to 0} \frac{e^{1+h} - e^1}{h} \approx \frac{e^{1.001} - e}{0.001} = 2.719$$

$$f(1) = e^1 = 2.718..$$

$$=$$

   (c) $f'(2)$

$$f'(2) = \lim_{h\to 0} \frac{e^{2+h} - e^2}{h} \approx \frac{e^{2.001} - e^2}{0.001} = 7.39...$$

$$f(2) = e^2 = 7.389..$$

$$=$$

   (d) $f'(-1)$

$$f'(-1) = \lim_{h\to 0} \frac{e^{-1+h} - e^{-1}}{h} \approx \frac{e^{0.999} - e^{-1}}{0.001} = 0.3680...$$

$$f(-1) = e^{-1} = 0.3678..$$

$$=$$

3. Derivative Rules for Exponential Functions

$$\frac{d}{dx}[e^x] = e^x$$

For $f(x) = e^x$,

$f'(x) = f(x)$.

y-values equal slope

$$y = a^x = e^{\ln(a^x)} = e^{x(\ln a)}$$

So, $y' = (\ln a) e^{x(\ln a)} = (\ln a) a^x$

$$\frac{d}{dx}\left[a^x\right] = (\ln a) a^x$$

4. Examples: Find the derivatives.

(a) $y = \overset{f}{x^4}\overset{g}{e^x}$

$y' = 4xe^x + x^4 e^x$

$f' \cdot g + f \cdot g'$

(b) $y = e^{x^2} = e^{(x^2)}$ ← chain rule!

$y' = \left(e^{x^2}\right)(2x) = 2x\,e^{x^2}$

(c) $y = 5^{-x} = 5^{(-x)}$ ← chain rule!

$y' = \left((\ln 5)\,5^{-x}\right)(-1)$

$= (-\ln 5)\,5^{-x}$

Alternate: $y = \left(\frac{1}{5}\right)^x$

$y' = \ln\left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^x$

(d) $f(x) = x^5 + 5^x$

$f'(x) = \underbrace{5x^4}_{\text{power rule}} + \underbrace{(\ln 5)5^x}_{\text{exponential rule}}$

Do you see why different rules are used?

5. Let $P(t) = P_0 e^{kt}$. Find $P'(t)$ and then write it in terms of $P(t)$.

$P'(t) = (P_0)(e^{kt})(k)$

$= P_0 k e^{kt}$

→ Rewrite in terms of $P(t)$

$P(t) = P_0 e^{kt}$

So $P'(t) = k\,P(t)$

OR $k = \dfrac{P'}{P}$ So k is the proportion of growth.

6. A population of bacteria has an initial population of 200 bacteria. The population is growing at a rate of 4 % per hour. → $P' = 0.04P$ or $k = 0.04$

(a) Write an exponential function $P(t)$ that relates the total population as a function of t where the units of t should be hours and the units of P should be number of bacteria.

$P(t) = 200\,e^{0.04t}$

Check: $P(0) = 200\,e^0 = 200$

$P(1) = 200\,e^{0.04} = 208.162$

$\dfrac{208-200}{200} = \dfrac{8}{200} = \dfrac{4}{100} = 4\%$

(b) Find and interpret $P'(1)$.

$P'(t) = (200)(0.04)\,e^{0.04t} = 8\,e^{0.04t}$

$P'(1) = 8\,e^{0.04} \approx 8.3264.$

At 1 hour, the population is increasing at a rate of 8 bacteria per hour.

(c) Find and interpret $P'(100)$.

$P(100) = 8\,e^4 = 436.7$ bacteria/hr

At 100 hours, the population is increasing at a rate of 436.7 bacteria per hour.

(d) Find $P'(1)/P(1)$ and $P'(100)/P(100)$. What do you observe?

$\dfrac{P'(1)}{P(1)} = \dfrac{8\,e^{0.04}}{200\,e^{0.04}} = \dfrac{1}{25} = 0.04$; $\dfrac{P'(100)}{P(100)} = \dfrac{8\,e^4}{200\,e^4} = \dfrac{1}{25} = 0.04.$

The rate of growth as a proportion of existing population is constant!