1. * The linear approximation, $L(x)$, of $f(x)$ at $x = a$ is:

$$L(x) = f(a) + f'(a)(x-a)$$

* Note: $L(x)$ is nothing more than the tangent line to $f(x)$ at $x=a$ written in a particularly useful way.

2. Let $f(x) = x^{4/3}$.

   (a) Find the linear approximation $L(x)$ of $f(x)$ at $a = 1$.

$f(x) = x^{4/3}$, $f(1) = 1^{4/3} = 1$

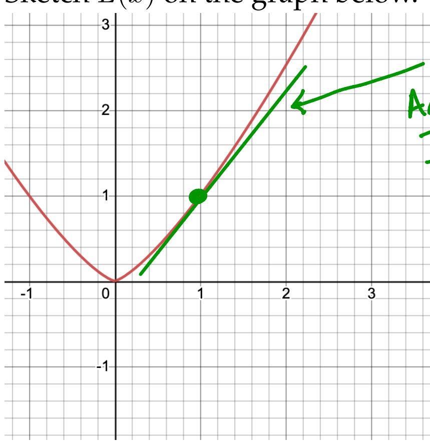
$f'(x) = \frac{4}{3}x^{1/3}$, $f'(1) = \frac{4}{3} \cdot 1^{1/3} = 4/3$

$L(x) = f(1) + f'(1)(x-1)$

$$\boxed{L(x) = 1 + \frac{4}{3}(x-1)}$$

Observe the one does NOT simply. The form is intentional.

   (b) Sketch $L(x)$ on the graph below.



$L(x)$: Again, it's just the tangent.

   (c) Use $L(x)$ to estimate $(1.1)^{4/3}$

$(1.1)^{4/3} \approx L(1.1) = 1 + \frac{4}{3}(1.1-1) = 1 + (1.33\overline{3})(0.1) = 1 + 0.133\overline{3}$

$= \boxed{1.133\overline{3}}$ estimate

   (d) Use your calculator to find $(1.1)^{4/3}$ exactly and determine the error between the exact value and the estimate.

using calculator: $(1.1)^{4/3} = 1.135508127\ldots$ exact value

error: $\underbrace{(1.1)^{4/3}}_{\text{exact value}} - \underbrace{L(1.1)}_{\text{estimated value}} = 0.00217479\ldots$ error (it's small!)

3. Estimate $\frac{1}{2.01}$ using an appropriate linear approximation (pick an $f(x)$ and an a). Use your calculator to determine the exact value.

Pick $f(x) = \frac{1}{x}$, $a = 2$.

$f(2) = \frac{1}{2}$

$f'(x) = -x^{-2}$

$f'(2) = \frac{-1}{4}$

$L(x) = \frac{1}{2} + \left(\frac{-1}{4}\right)(x-2) = 0.5 - 0.25(x-2)$

$\frac{1}{2.01} \approx L(2.01) = 0.5 - 0.25(2.01-2) = 0.5 - 0.25(0.01)$

$= 0.5 - 0.0025$

$= 0.4975$

Calculator: $\frac{1}{2.01} = 0.49751243\ldots$

dy - an estimated change in y

f'(x) - the tangent-line-estimation of how much y changes given a 1-unit change in x

dx - how much x actually changed

4. The differential of $y = f(x)$ is $\boxed{dy = f'(x)\,dx}$

* just the derivative written a different way.

5. Given $f(x) = x\sin(\frac{\pi}{2}x)$.

(a) Find the differential of $f(x)$ and evaluate the differential when $x = 2$ and $dx = 0.1$.

differential: $dy = \left[1 \cdot \sin(\tfrac{\pi}{2}x) + x \cdot \cos(\tfrac{\pi}{2}x) \cdot \tfrac{\pi}{2}\right] dx$

$$\boxed{dy = \left[\sin(\tfrac{\pi}{2}x) + \frac{\pi x}{2}\cos(\tfrac{\pi}{2}x)\right] dx}$$

evaluate: $dy = \left(\sin(\tfrac{\pi}{2}\cdot 2) + \frac{\pi \cdot 2}{2}\cos(\tfrac{\pi}{2}\cdot 2)\right)(0.1) = (0 + \pi(-1))(0.1) = -0.1\pi$

$= \boxed{-0.31415\ldots}$

(b) Use a calculator to find $f(2.1) - f(2)$.

$$f(2.1) - f(2) = \left[(2.1)\sin\left(\tfrac{\pi}{2}(2.1)\right)\right] - \left[2\underbrace{\sin\left(\tfrac{\pi}{2}\cdot 2\right)}_{=0}\right] = -0.3285123\ldots$$

(c) Explain what the calculations in parts (a) and (b) represent and why they are close but not the same.

Part (b) calculates exactly how much y changes when x changes from x=2 to x=2.1.

Part (a) estimates how much y will change when x changes from x=2 to x=2.1 using the tangent line.

6. The side of a cube is measured to be 2 meters with a possible error in measurement of 0.1 meter. Use differentials to estimate the maximum possible error when computing the volume of the cube. Determine the relative error.

- $V = s^3$

$dV = 3s^2\,ds$

- $s = 2$

$ds = 0.1$

maximum error $\approx dV = 3(2)^2(0.1) = 1.2\ m^3$

relative error $\approx \frac{dV}{V} = \frac{1.2}{2^3} = \frac{1.2}{8} = 0.15$

or 15%