Section 5.5: Substitution (i.e. undoing the Chain Rule)

Goals: (i) Practice u-substitution (ii) Practice *sophisticated* u-substitution (iii) Practice substitution with both indefinite and definite integrals (iv) Develop intuition about how to choose u.

1. (a) Verify that the formula is correct: $\int \frac{2x}{\sqrt{x^2-1}}\,dx = 2\sqrt{x^2-1}+C$

Let $y = 2(x^2-1)^{1/2}+C$

Then $y' = 2\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x) = \boxed{\frac{2x}{\sqrt{x^2-1}}}$ ✓

So, yes!, the formula is correct.

(b) Use the substitution $u = x^2-1$ to rewrite *the entire integral* in terms of u. Then integrate the integral with the new variables.

$u = x^2-1$

$du = 2x\,dx$

$\int \frac{2x}{\sqrt{x^2-1}}\,dx = \int (x^2-1)^{-1/2}(2x\,dx) \overset{\text{substitute}}{=} \int u^{-1/2}\,du \overset{\text{integrate}}{=} 2u^{1/2}+C$

$= 2(x^2-1)^{1/2}+C$

resubstitute.

2. Explain why the formula is not correct: $\int \sqrt{x^2+1}\,dx = \frac{1}{3}(x^2+1)^{3/2}+C$

$y = \frac{1}{3}(x^2+1)^{3/2}+C$

$y' = \frac{1}{3}\cdot\frac{3}{2}(x^2+1)^{\frac{1}{2}}(2x) = x\sqrt{x^2+1}$

These are NOT equal. So the formula is NOT correct.

3. $\int t^3\cos(t^4+1)\,dt = \int [\cos(t^4+1)]\cdot[t^3\,dt] = \int \cos(u)\cdot\left(\frac{1}{4}du\right) = \frac{1}{4}\int \cos(u)\,du$

let $u = t^4+1$

$du = 4t^3\,dt$

$\frac{1}{4}du = t^3\,dt$

$= \frac{1}{4}\sin(u)+C = \frac{1}{4}(t^4+1)+C$

4. $\int 5\sin^2(x)\cos(x)\,dx = 5\int [\sin(x)]^2[\cos(x)\,dx] = 5\int u^2\,du = \frac{5}{3}u^3+C$

let $u = \sin(x)$

$du = \cos(x)\,dx$

$= \frac{5}{3}(\sin(x))^3+C$

5. $\int \frac{dx}{(8-5x)^3} = \int (8-5x)^{-3}\,dx = \int u^{-3}\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int u^{-3}du = -\frac{1}{5}\left(-\frac{1}{2}u^{-2}\right)+C$

$= \frac{1}{10}(8-5x)^{-2}+C$

let $u = 8-5x$

$du = -5dx$

$-\frac{1}{5}du = dx$

6. $\int \sin\left(\frac{\pi x}{4}\right) dx = \frac{4}{\pi}\int \sin(u)\,du = -\frac{4}{\pi}\cos(u)+C$

let $u = \frac{\pi}{4}x$

$du = \frac{\pi}{4}dx$

$\frac{4}{\pi}du = dx$

$= -\frac{4}{\pi}\cos\left(\frac{\pi}{4}x\right)+C$

7. $\int_0^1 (x-1)(x^2-2x)^{10}\,dx = \int_0^1 (x^2-2x)^{10}((x-1)dx) = \int_0^{-1} u^{10}\left(\frac{1}{2}du\right) = \frac{1}{2}\int_0^{-1} u^{10}du$

let $u = x^2-2x$

$du = (2x-2)dx$

$\frac{1}{2}du = (x-1)dx$

If $x=0,\ u=0$

$x=1,\ u=-1$

$= \frac{1}{2}\cdot\frac{1}{11}u^{11}\Big]_0^1 = \frac{1}{22}(1^{11}-0^{11})$

$\boxed{= \frac{1}{22}}$

8. $\int_0^{\pi/4} \tan^3(\theta)\sec^2(\theta)\,d\theta = \int_0^{\frac{\pi}{2}} (\tan\theta)^3(\sec^2\theta\,d\theta) = \int_0^1 u^3du$

let $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

If $\theta=0,\ u=\tan 0=0$

$\theta=\frac{\pi}{4},\ u=\tan\left(\frac{\pi}{4}\right)=1$

$= \frac{1}{4}u^4\Big]_0^1 = \boxed{\frac{1}{4}}$

9. $\int (x^4-5)^{1/3}x^7\,dx = \int \underline{(x^4-5)}^{\frac{1}{3}}\cdot \underline{x^4}\cdot(\underline{x^3dx}) = \int (u^{\frac{1}{3}})(u+5)(du) = \int \left(u^{\frac{4}{3}}+5u^{\frac{1}{3}}\right)du$

let $u = x^4-5$

$du = 4x^3dx$

$\frac{1}{4}du = x^3dx$

$u+5 = x^4$

$= \frac{3}{7}u^{\frac{7}{3}} + 5\cdot\frac{3}{4}u^{\frac{4}{3}}+C$

$\boxed{= \frac{3}{7}(x^4-5)^{7/3} + \frac{15}{4}(x^4-5)^{\frac{4}{3}}+C}$