Section 2-2: The Limit of a Function

Read Section 2.2. Work the embedded problems.
Goals:

- Understand the meaning of the notation $\lim\limits_{x\to a} f(x) = L$.
- Know how to evaluate one- and two-sided limits both from a graph and numerically.
- Understand the relationship between infinite limits and vertical asymptotes.

1. $\boxed{\text{DEFINITION:}}$ **two-sided limit**

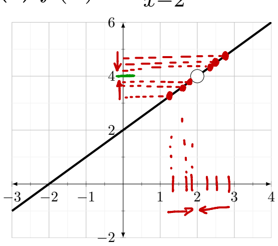
   Notation: $\lim\limits_{x\to a} f(x) = L$ ("a" and "L" are numbers)

   Words: the limit of f(x), as x approaches a, is L.

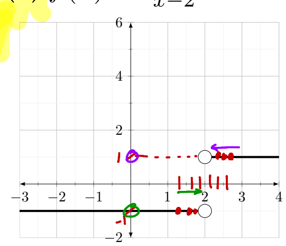
   It means: as x's get closer + closer to the x-value a (both from above and from below) give y-values that get closer + closer to L when those x's are plugged into f(x). Note, the limit does not care what happens when x=a, only when x is closer + closer to it.

2. Evaluate the limits below using the graph and confirm your answers numerically.

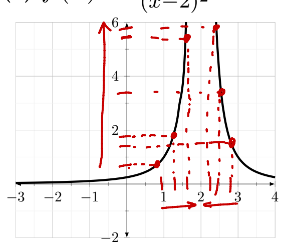
(a) $f(x) = \frac{x^2-4}{x-2}$



(b) $f(x) = \frac{|x-2|}{x-2}$



(c) $f(x) = \frac{1}{(x-2)^2}$



graphically:

$\lim\limits_{x\to 2} \frac{x^2-4}{x-2} = 4$

$\lim\limits_{x\to 2} \frac{|x-2|}{x-2} = \text{DNE}$ (Does not exist.)

Because one side gets close to −1 and one side gets close to +1.

$\lim\limits_{x\to 2} \frac{1}{(x-2)^2} = \infty$

As x→2, the y-values grow without bound.

(Note: This limit is also DNE; but "∞" is a more complete answer than "DNE".)

numerically:

X	$\frac{x^2-4}{x-2}$
1.5	3.5
1.9	3.9
1.99	3.99
1.999	3.999
2	DNE
2.001	4.001
2.01	4.01
2.1	4.1
2.5	4.5

as x→2 f(x)→4

X	$\lvert x-2\rvert/(x-2)$
1.5	−1
1.9	−1
1.99	−1
1.999	−1
2	DNE
2.001	+1
2.01	+1
2.1	+1
2.5	+1

as x→2

f(x) goes to −1

f(x) goes to +1

X	$1/(x-2)^2$
1.5	4
1.9	100
1.99	10,000
1.999	1,000,000
2	DNE
2.001	1,000,000
2.01	10,000
2.1	100
2.5	4

f(x) blows up to +∞

3. Numerically or graphically, determine the limits below. Assume a and c are fixed constants.

(a) $\lim\limits_{x\to 0} = \dfrac{\sin(x)}{x}$ $= 1$

x	-0.5	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.5
$\frac{\sin(x)}{x}$	0.95885	0.99833416...	0.9999833...	0.9999998...	DNE	0.9999998...	0.9999833...	0.99833416...	0.958851...

(b) $\lim\limits_{x\to 1} 5 =$ 5

x	0.5	.99	1	1.01	1.5
5	5	5	5	5	5

always 5

(c) $\lim\limits_{x\to 2} 5 =$ 5

same!

(d) $\lim\limits_{x\to a} c =$ c

always c.

(e) $\lim\limits_{x\to 1} x =$ 1

x	0.5	0.99	1	1.01	1.5
x	0.5	0.99	1	1.01	1.5

(f) $\lim\limits_{x\to 2} x =$ 2

x	1.5	1.99	2	2.01	2.1
x	1.5	1.99	2	2.01	2.1

(g) $\lim\limits_{x\to a} x =$ a

4. Return to problem 2b above. Evaluate the limits below assuming that

$x \to 2^-$ means X approaches 2 from the left (or from x-values a little smaller than a)

and

$x \to 2^+$ means X approaches 2 from the right (or from x-values a little larger than a)

(a) $\lim\limits_{x\to 2^-} \dfrac{|x-2|}{x-2} =$ -1

(b) $\lim\limits_{x\to 2^+} \dfrac{|x-2|}{x-2} =$ $+1$

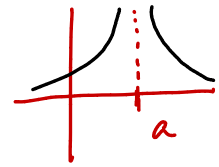
See *

5. What must be the relationship between the existence of two-sided limits in terms of one-sided limits?

(notation) $\boxed{\lim\limits_{x\to a} f(x) = L}$ is equivalent to $\boxed{\lim\limits_{x\to a^+} f(x) = L = \lim\limits_{x\to a^-} f(x)}$

(words) The two sided limit is L

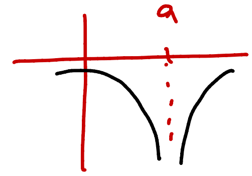
Each one-sided limit exists and is equal to the same number, L.



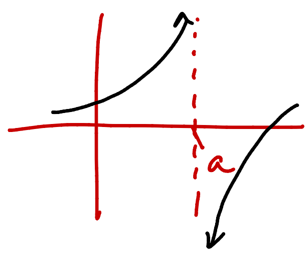
6. $\boxed{\text{DEFINITION:}}$ infinite limits

- $\lim_{x\to a} f(x)=\infty$ means f(x) grows w/o bound as x approaches x (on both sides)

- $\lim_{x\to a} f(x)=-\infty$ means f(x) decreases w/o bound as x approaches x (on both sides)



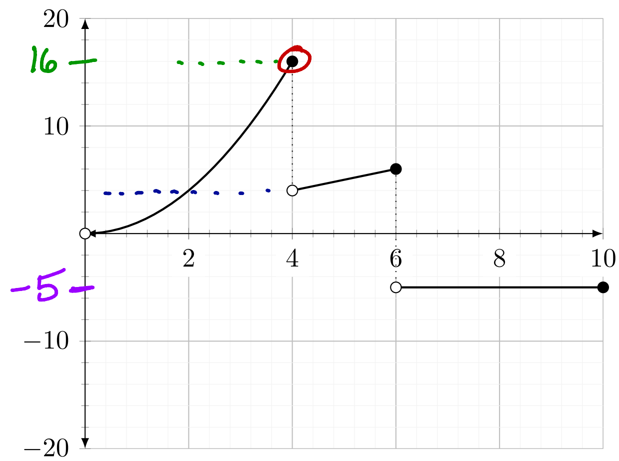
- These can also be one-sided!



$\lim_{x\to a^-} f(x)=+\infty$

$\lim_{x\to a^+} f(x)=-\infty$

7. The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} f(x) =$ 16

(b) $\lim_{x\to 4^+} f(x) =$ 4

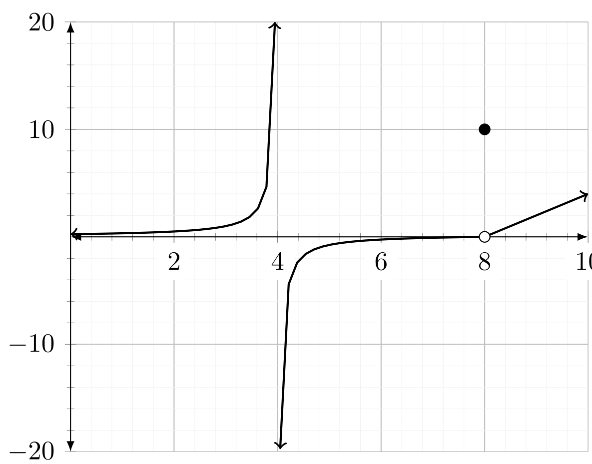
(c) $\lim_{x\to 4} f(x) =$ DNE

(d) $f(4) =$ 16

(e) $\lim_{x\to 8} f(x) =$ -5

(f) $f(8) =$ -5

8. The function $g(x)$ is graphed below. Use the graph to fill in the blanks.



(a) $\lim_{x\to 4^-} f(x) =$ $+\infty$

(b) $\lim_{x\to 4^+} f(x) =$ $-\infty$

(c) $\lim_{x\to 4} f(x) =$ DNE

(d) $f(4) =$ DNE

(e) $\lim_{x\to 8} f(x) =$ 0

(f) $f(8) =$ 10

9. Find any vertical asymptotes of $f(x) = \frac{2}{x+5}$ and *justify* your answer using a limit.

V.a.: $x = -5$ ← Value makes denominator zero.

Justification:

$$\lim_{x \to -5^+} \frac{2}{x+5} = +\infty$$

as $x \to -5^+$ (#'s like $-4.9, -4.99$)

$x + 5 \to 0^+$

10. Sketch the graph of an function that satisfies *all* of the given conditions. Compare your answer with that of your neighbor.

$\lim_{x \to 0^-} f(x) = 1$ $\quad \lim_{x \to 0^+} f(x) = -2$ $\quad \lim_{x \to 4^-} f(x) = 3$ $\quad \lim_{x \to 4^+} f(x) = 0$

$f(0) = -2$ $\qquad f(4) = 1$