goals:

- Know how to evaluate limits algebraically (that is, using the limit laws from this section)
- Recognize when a limit needs some algebraic manipulation and when it doesn't.
- Understand the idea behind the Squeeze Theorem.

Recall that in the Section 2.2 notes we established $\boxed{\lim_{x\to 0} \frac{\sin(x)}{x} = 1.}$

Rule: Example

1. $\lim_{x\to a} c = c$ — $\lim_{x\to 5} 14 =$ 14

2. $\lim_{x\to a} x = a$ — $\lim_{x\to 5} x =$ 5

3. $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f + \lim_{x\to a} g(x)$ — $\lim_{x\to 0} \frac{\sin(x)}{x} + (2x+\sqrt{2}) =$ $1+\sqrt{2}$

4. $\lim_{x\to a} f(x)-g(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ — $\lim_{x\to 0}\lim_{x\to 0} \frac{\sin(x)}{x} - (2x+\sqrt{2}) =$ $1-\sqrt{2}$

5. $\lim_{x\to a} c f(x) = c \lim_{x\to a} f(x)$ — $\lim_{x\to 0}\lim_{x\to 0} \frac{35\sin(x)}{x} =$ $35\cdot 1 = 35$

6. $\lim_{x\to a} (f(x))(g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$ — $\lim_{x\to 4} (5x+20)(x-2) =$ $(20+20)(4-2) = 40(2) = 80$

7. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim_{x\to a} g(x)}$ provided it's not zero! — $\lim_{x\to 4} \frac{5x+20}{x-2} =$ $\frac{40}{2} = 20$

8. $\lim_{x\to a} (f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$ — $\lim_{x\to -2} (8+5x)^5 =$ $(8-10)^5 = (-2)^5 = -32$

9. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ — $\lim_{x\to -1} \sqrt{15-x} =$ $\sqrt{16} = 4$

Nutshell: You can evaluate complex limits by finding the limits of each piece separately provided no expression is undefined!

1. lesson: Alway try plugging in first.

$$\lim_{x\to\sqrt{2}} 5x - \sqrt{8x^2-1}$$

$$= 5\sqrt{2} - \sqrt{8(\sqrt{2})^2 - 1} = 5\sqrt{2} - \sqrt{8\cdot 2 - 1}$$

$$= 5\sqrt{2} - \sqrt{15}$$

2. lesson: Problem: Denominator is zero. Solution Factor + Cancel

$$\lim_{t\to 2} \frac{x^2-4}{x-2} = \frac{2^2 - 4}{2-2} = \frac{0}{0} \leftarrow !!$$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2 = 2+2 = 4$$

Bonus: Is this fair? Why?

3. lesson: Problem: Denominator is zero! Solution: Rationalize!

$$\lim_{x\to 5} \frac{3-\sqrt{x+4}}{5-x} = \frac{3-\sqrt{9}}{5-5} = \frac{0}{0} \leftarrow !!$$

$$\lim_{x\to 5} \frac{(3-\sqrt{x+4})}{(5-x)} \cdot \frac{(3+\sqrt{x+4})}{(3+\sqrt{x+4})} = \lim_{x\to 5} \frac{9-(x+4)}{(5-x)(3+\sqrt{x+4})} = \lim_{x\to 5} \frac{5-x}{(5-x)(3+\sqrt{x+4})}$$

$$= \lim_{x\to 5} \frac{1}{3+\sqrt{x+4}} = \frac{1}{3+\sqrt{9}} = \frac{1}{6}$$

4. lesson: Problem: Denominator is zero. Solution: Combine fractions

$$\lim_{x\to 2} \frac{\frac{1}{4}-\frac{1}{2+x}}{x-2} = \frac{\frac{1}{4}-\frac{1}{4}}{2-2} = \frac{0}{0} \;!!$$

$$\lim_{x\to 2} \frac{\frac{1}{4}-\frac{1}{2+x}}{x-2} = \lim_{x\to 2} \frac{\frac{((2+x)-4)}{4(2+x)}}{(x-2)} = \lim_{x\to 2} \frac{\frac{x-2}{4(2+x)}}{(x-2)}$$

$$= \lim_{x\to 2} \frac{1}{4(2+x)} = \frac{1}{16}$$

5. lesson: Problem: Denominator is zero AND numerator is not!

Solution: limit must blow-up.

$$\lim_{x\to 2^-} \frac{x^2+4}{x-2} = \frac{8}{0}$$

as $x \to 2^-$ (#'s like 1.9, 1.99), $x-2 \to 0^-$ and $x^2+4 \to 8$

So $\frac{+}{-} = -$. So $\lim_{x\to 2^-} \frac{x^2+4}{x-2} = -\infty$.

6. The last two problems reference the function $f(x) = \begin{cases} \frac{1}{2x} & \text{if } 0 < x \le 2 \\ 0 & \text{if } 2 < x \end{cases}$

(a) $\lim_{x\to 2} f(x)$ = DNE

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1}{2x} = \frac{1}{4}$.

The l.h. limit and the r.h. limit are not equal!

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 0 = 0$.

(b) $\lim_{x\to 2^+} e^{f(x)} = e^{\lim_{x\to 2^+} f(x)} = e^{\frac{1}{4}}$

7. Squeeze Theorem

Know

- $f(x) \le \boxed{g(x)} \le h(x)$, close to a
- $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Conclude

$\lim_{x \to a} g(x) = L$

Idea