Your Name

Your Signature

Solutions.

Problem	Total Points	Score
1	6	
2	8	
3	6	
4	6	
5	8	
6	10	
7	10	
8	10	
9	10	
10	6	
11	10	
12	10	
extra credit	5	
Total	100	

- You have 1 hour to complete the midterm.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.
- The last page of the exam contains formulas.

1. (6 points) Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{k}$.

   (a) Find a unit vector that is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -\vec{i} - (-2)\vec{j} + (-1)\vec{k} = \langle -1, 2, -1 \rangle$$

$$1^2 + 2^2 + 1^2 = 6$$

$$\boxed{\vec{u} = \frac{1}{\sqrt{6}} \langle -1, 2, -1 \rangle}$$

   (b) Find the vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}}\mathbf{b}$.

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = \left(\frac{\langle 2,1,0 \rangle \cdot \langle 1,0,-1 \rangle}{2^2 + 1^2} \right) \left(\langle 2,1,0 \rangle \right)$$

$$= \boxed{\frac{2}{5} \langle 2,1,0 \rangle}$$

2. (8 points) Find an equation of the plane through the point $(1, 2, -1)$ and parallel to the plane $4x - y - 3z = 6$. Simplify to the form $ax + by + cz + d = 0$.

$$\vec{n} = \langle 4, -1, -3 \rangle$$

$$4(x-1) - 1(y-2) - 3(z+1) = 0$$

$$4x - 4 - y + 2 - 3z - 3 = 0$$

$$\boxed{4x - y - 3z - 5 = 0}$$

3. (6 points) Find the parametric equations for the tangent line to the curve

$\vec{r}(t) = t^2\vec{i} + \sin t\,\vec{j} + \cos t\,\vec{k}$ when $t = \pi/2$.

So $\vec{r}(\frac{\pi}{2}) = \langle \frac{\pi^2}{4}, 1, 0 \rangle$; $\vec{r}\,'(t) = \langle 2t, \cos t, -\sin t \rangle$, $\vec{r}\,'(\pi/2) = \langle \pi, 0, -1 \rangle$

answer: line is: $x = \frac{\pi^2}{4} + \pi t$

$y = 1$

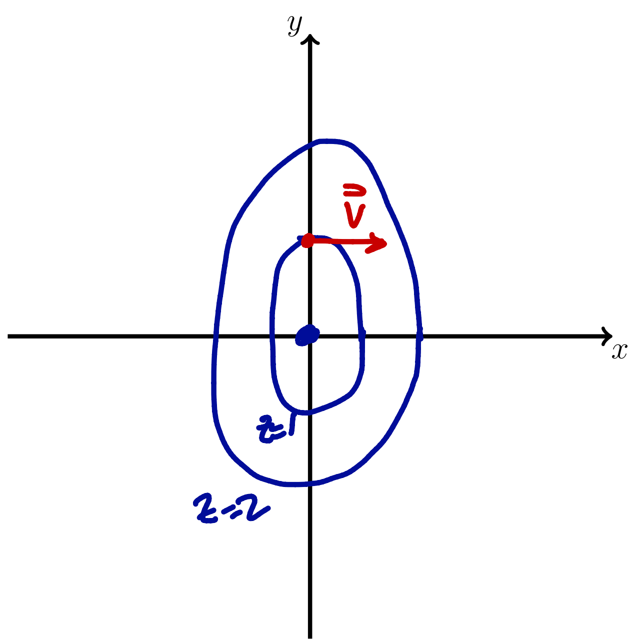
$z = -t$

4. (6 points) For the surface $f(x,y) = \sqrt{4x^2 + y^2}$, sketch and label the level curves for $z = 0$, $z = 1$, and $z = 2$ on the axes below. Add to your graph, a vector $\mathbf{v}$ such that the derivative of $f(x,y)$ at the point $(0,1)$ in the direction of $\mathbf{v}$ is zero.

$z=0$: $0 = 4x^2 + y^2$ $(0,0)$

$z=1$: $1 = 4x^2 + y^2$ ellipse

$z=2$: $4 = 4x^2 + y^2$ bigger ellipse



5. (8 points) Find ~~and simplify~~ the linearization, $L(x,y)$, of the function $f(x,y) = 5y\sqrt{x}$ at the point $(1,4)$.

$f_x = \frac{5}{2} y x^{-1/2}$, $f_x(1,4) = \frac{5}{2} \cdot 4 = 10$

$f_y = 5\sqrt{x}$, $f_y(1,4) = 5$

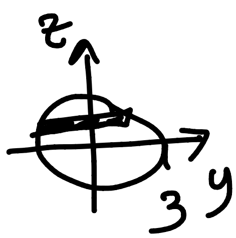
$f(1,4) = 10.$

$z - 10 = 10(x-1) + 5(y-4)$

$\boxed{L(x,y) = 10 + 10(x-1) + 5(y-4)}$

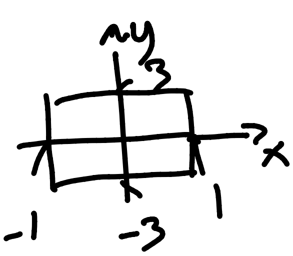
6. (10 points) Let E be the solid bounded by the surfaces $y^2 + z^2 = 9$, $x = -1$, and $x = 1$. Set up, but do not evaluate, the triple integral $\iiint_E f(x,y,z)\,dV$ using the given order of integration:

(a) $dx\,dy\,dz$



$$\int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{-1}^{1} f(x,y,z)\,dx\,dy\,dz$$

(b) $dz\,dy\,dx$



$$\int_{-1}^{1} \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y,z)\,dz\,dy\,dx$$

7. (10 points)

(a) Find all critical points of the function:

$$f(x, y) = 2 - x^4 + 2x^2 - y^2$$

$f_x = -4x^3 + 4x = -4x(x^2-1) = 0, \ x = 0, \pm 1$

$f_y = -2y = 0, \ y = 0$

points: $(0,0), (1,0)\ (-1,0)$

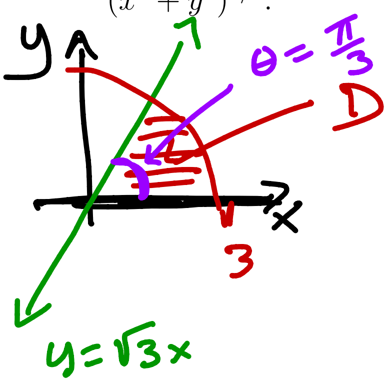
(b) Find all local maxima, local minima, and saddle points of the function in part **(a)**.

$D = f_{xx}f_{yy} - (f_{xy})^2 = (-12x^2+4)(-2) - 0 = -8(3x^2-1)$

point	D	f_{xx}	conclusion
$(0,0)$	>0	$+ \cup$	local min
$(1,0)$	<0	$\sim$	saddle
$(-1,0)$	<0	$\sim$	saddle

rewrite

8. (10 points) Find the mass of the lamina that occupies the region D in the first quadrant bounded by $y = 0$, $y = \sqrt{3}\,x$, and $x^2 + y^2 = 9$ and has density function $\rho(x, y) = (x^2 + y^2)^{3/2}$.



$$m = \iint_D \rho \, dA = \int_0^{\frac{\pi}{3}} \int_0^3 (r^2)^{3/2} \, r \, dr \, d\theta$$

$$= \left(\int_0^{\pi/3} d\theta\right)\left(\int_0^3 r^4 \, dr\right)$$

$$= \frac{\pi}{3}\left(\frac{1}{5} r^5\right) = \frac{81\pi}{3}$$

9. (10 points) Find the volume of the solid which is above the cone $z = \sqrt{x^2 + y^2}$ and inside (below) the sphere $x^2 + y^2 + z^2 = 1$.

$x^2 + y^2 + z^2 = 0$ or $\rho = 1$

$z = \sqrt{x^2 + y^2}$ or $z = r$ or $\phi = \frac{\pi}{4}$

$$V = \iiint_E 1 \, dV = \int \int \int \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \cdot \left(\int_0^{\pi/4} \sin\phi \, d\phi \right) \left(\int_0^1 \rho^2 \, d\rho \right)$$

$$= 2\pi \left((-\cos\phi) \Big|_0^{\pi/4} \right) \left(\frac{1}{3} \rho^3 \Big|_0^1 \right)$$

$$= 2\pi \left(-\frac{\sqrt{2}}{2} + 1 \right) \left(\frac{1}{3} \right) = \frac{(2 - \sqrt{2})\pi}{3}$$

10. (6 points) Match the vector fields $\mathbf{F}$ with the plots labeled I,II,III,IV:

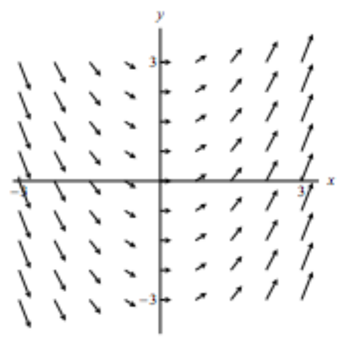
(a) $\mathbf{F}(x, y) = \langle x, 1\rangle$ III ⟵ *write "I", "II",... in the spaces*

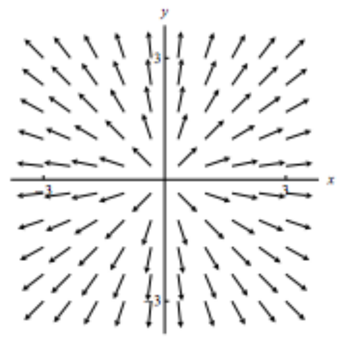
(b) $\mathbf{F}(x, y) = \langle 1, x\rangle$ I

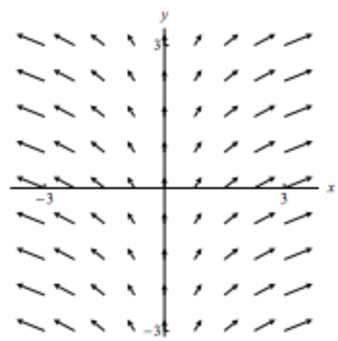
(c) $\mathbf{F}(x, y) = \nabla f$ IV

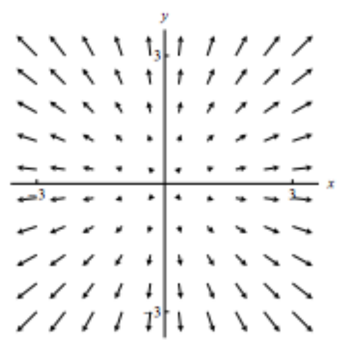
where $f(x, y) = x^2 + y^2$

(d) $\mathbf{F}(x, y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$ II


(I)


(II)


(III)


(IV)

check dir.

11. (10 points) Find the work done by the force field $\mathbf{F}(x,y) = (1+xy)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$ on an object moving along the line segment from (-1, 2) to (3, 0).

$(0,2)$ ✗

$$W = \int_C \vec{F}\cdot d\vec{r} = f(3,0) - f(0,2)$$

$$= \left(3\cdot e^0 - (-1)e^2\right)$$

$$= 3 + e^2$$

$$F = \langle \underbrace{(1+xy)e^{xy}}_{P},\ \underbrace{x^2e^{xy}}_{Q} \rangle$$

$$P_y = xe^{xy} + (1+xy)(x)e^{xy}$$
$$= e^{xy}(2x + x^2y)$$

equal.

$$Q_x = 2xe^{xy} + x^2\cdot e^{xy}\cdot y$$
$$= e^{xy}(2x + x^2y)$$

potential: $f(x,y) = xe^{xy} + K$

check: $f_x = 1\cdot e^{xy} + xye^{xy}$ ✓

$f_y = x^2e^{xy}$ ✓

apply the Fundamental Thm of Line Integrals

12. (10 points) Suppose $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for *any* positively-oriented, closed, simple curve C enclosing a region D which has area 10. (*Hint.* Green's Theorem)

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \underbrace{(x^2+y)}_{P}\, dx + \underbrace{(3x-y^2)}_{Q}\, dy = \iint_D \big(\underbrace{3}_{Q_x} - \underbrace{1}_{P_y}\big)\, dA$$

$$= 2 \iint_D dA = 2 \cdot 10 = \underline{\underline{20}}.$$

Extra Credit (5 points) Assume $\mathbf{F}(x, y)$ is a conservative vector field defined on a open, connected region D. Fix any point (a, b) in D. ① Explain why the formula

$$f(x, y) = \int_{(a,b)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

defines a function f on D. ② Explain why this function satisfies $\nabla f = \mathbf{F}$.

① Because $\vec{F}$ is conservative, $\int_C \vec{F} \cdot d\vec{r}$ is path independant.

So for every point, (x,y), $\int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}$ is uniquely defined.

So $f(x,y)$ is a function

②. $f_x(x,y) = \frac{\partial}{\partial x}\left(\int_{(a,b)}^{(x,y)} P\,dx + Q\,dy \right) = \frac{\partial}{\partial x}\left(\int_{(a,b)}^{(x,y)} P\,dx \right) = P$

since $dy = 0$.

Similarly $f_y = Q$.

So $\nabla f = \langle P, Q \rangle = \vec{F}$.

2:52