Your Name

Solutions

Your Signature

Problem	Total Points	Score
1	10	
2	15	
3	10	
4	15	
5	10	
6	15	
7	10	
8	15	
extra credit	5	
Total	100	

- You have 1 hour to complete the midterm.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.

1. (10 points) Evaluate the triple integral:

$$\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y \sin(x)\, dz\, dy\, dx$$

$$= \int_0^\pi \int_0^1 (\sin x)(1-y^2)^{1/2}(y)\, dy\, dx$$

$$= \left(\int_0^\pi \sin x\, dx\right)\left(\int_0^1 (1-y^2)^{1/2} y\, dy\right)$$

$$= \left(-\cos x\right]_0^\pi \cdot \left(-\frac{1}{3}(1-y^2)^{3/2}\right]_0^1$$

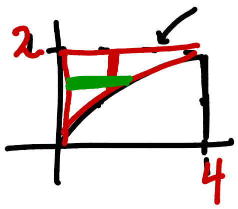
$$= (1+1)\left(-\frac{1}{3}\right)(0-1) = \frac{2}{3}$$

2. (15 points) Evaluate the double integral:

$$\int_0^4 \int_{\sqrt{x}}^2 \sqrt{y^3+1}\,dy\,dx$$

$0 \leq x \leq 1$

$\sqrt{x} \leq y \leq 1$



$y = \sqrt{x}$ or

$y^2 = x$

$$= \int_0^2 \int_0^{y^2} (y^3+1)^{1/2}\,dx\,dy$$

$$= \int_0^2 (y^3+1)^{1/2}\,y^2\,dy$$

$$= \frac{2}{9}(y^3+1)^{3/2}\Big]_0^2$$

$$= \frac{2}{9}\left(9^{3/2}-1\right) = \frac{2}{9}(27-1) = \frac{2\cdot 26}{9} = \frac{52}{9}$$

3. (10 points) Change the integral below into cylindrical coordinates:

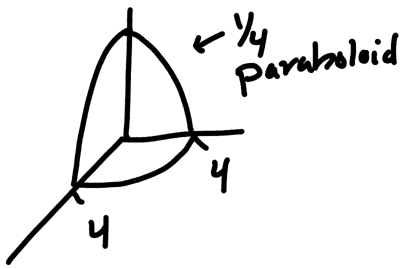
$$\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz\, dy\, dx$$

$0 \le z \le 16-x^2-y^2$

$0 \le y \le \sqrt{16-x^2}$

$0 \le x \le 4$

$$= \int_0^{\pi/2} \int_0^4 \int_0^{16-r^2} r \cdot r \, dz\, dr\, d\theta$$



4. (15 points) Let E be the solid that lies above the xy-plane and below the paraboloid $z = 2 - x^2 - y^2$. Assume E has density function $\rho(x, y, z) = x^2 + z^4$.

(a) Set up but do not evaluate the integral expression for the mass, m, of the solid.

$$m = \iiint_E \rho \, dV = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_0^{2-x^2-y^2} (x^2+z^4)\, dz\, dy\, dx$$

xy-plane : $z=0$. or

$0 = 2-x^2-y^2$ or

$x^2+y^2=2$

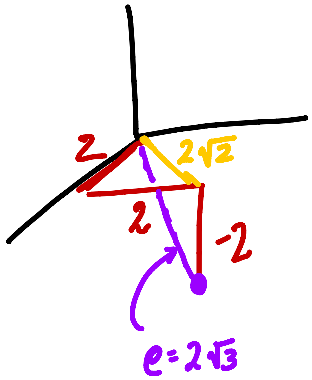
(b) Set up but do not evaluate the expression for $\overline{z}$, the z-coordinate of the center of mass of E.

$$M_{xy} = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2+x^2}} \int_0^{2-x^2-y^2} z(x^2+z^4)\, dz\, dy\, dx$$

$$\overline{z} = \frac{M_{xy}}{m}$$

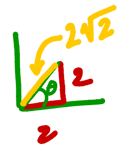
5. (5 points each)

(a) Convert the point $(x, y, z) = (2, 2, -2)$ in rectangular coordinates to spherical coordinates.

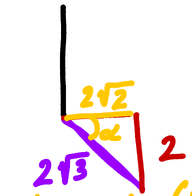


• $\rho = \sqrt{4+4+4} = \underline{\underline{2\sqrt{3}}}$

$\theta = \pi/4$



So $r = 2\sqrt{2}$.



$\alpha = \arctan\left(\frac{2}{2\sqrt{2}}\right) = \arctan\left(\frac{1}{\sqrt{2}}\right)$

<u>Ans</u>: $(\rho, \theta, \phi) = \left(2\sqrt{3}, \frac{\pi}{4}, \frac{\pi}{2} + \arctan\left(\frac{2}{2\sqrt{2}}\right)\right)$

Note: $\phi = \frac{\pi}{2} + \arctan\left(\frac{1}{\sqrt{2}}\right) = \arccos\left(\frac{-1}{\sqrt{3}}\right) = \pi - \arctan(\sqrt{2}) = \frac{\pi}{2} + \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(b) Describe in words, with a picture if you like, the region described by the inequalities:

$$1 \leq \rho \leq 3, \quad 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$$

~~A Solid.~~
~~Take the top half of a solid ball of radius 3 centered at the origin + scoop out a solid ball of radius 1 centered at the origin. What is left is the solid.~~

6. (15 points) Rewrite the integral below in *spherical* coordinates. The expressions in your answer must be simplified.

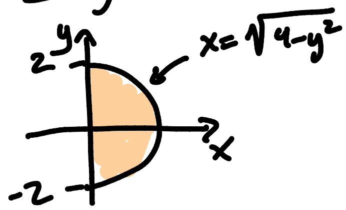
$$\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$$

$z = \sqrt{4-x^2-y^2}$ or

$z^2 + x^2 + y^2 = 4$

$0 \leq x \leq \sqrt{4-y^2}$

$-2 \leq y \leq 2$



$$\underset{\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}} \underset{\phi}{\int_{0}^{\pi}} \underset{\rho}{\int_{0}^{2}} \rho^2 \sin^2\phi \sin^2\theta \cdot \rho \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

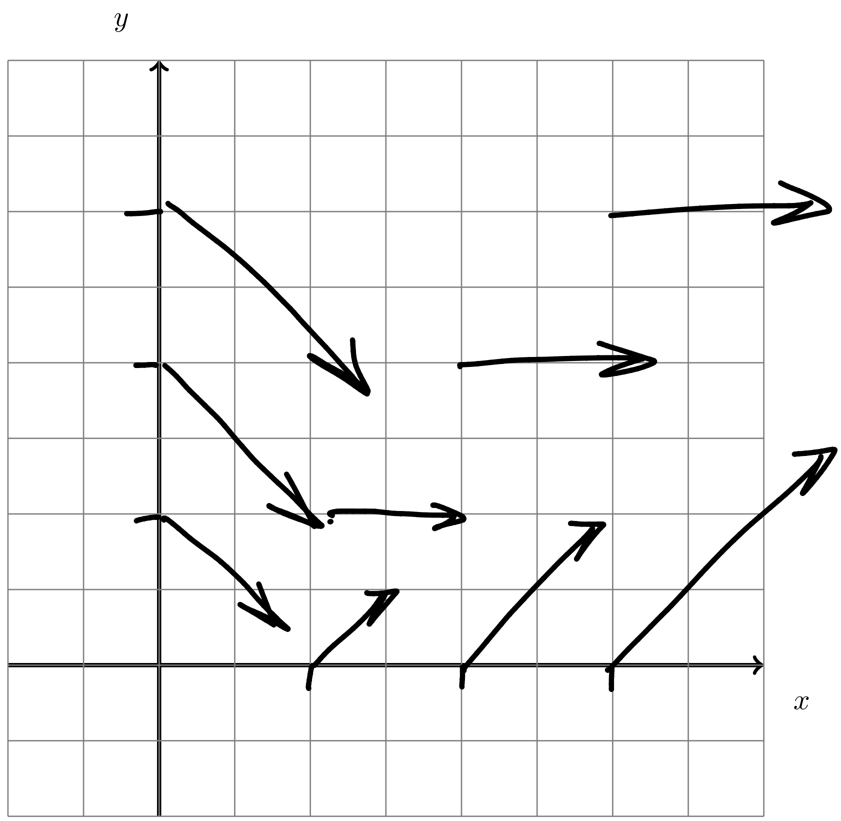
$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{\pi/2} \int_{0}^{2} \rho^5 \sin^3\phi \sin^2\theta \, d\rho\, d\phi\, d\theta$$

7. (10 points) On the graph below, make a rough sketch of the vector field

$$\mathbf{F}(x,y) = (x+y)\mathbf{i} + (x-y)\mathbf{j}$$

Your sketch does not have to be perfect or to scale, but your vectors should be in roughly the correct direction and *relative* length. Make sure to include vectors at:

- several points along the positive x-axis
- several points along the postive y-axis
- the points $(1,1)$, ~~$(2,1)$~~, ~~$(3,1)$~~ $(3,3)$
- the points ~~$(1,2)$~~, $(2,2)$, ~~$(3,2)$~~



on x-axis: $y=0$

$F = \langle x, x\rangle$

on y-axis: $x=0$

$F = \langle y, -y\rangle$

pt	vector
(1,1)	$\langle 2,0\rangle$
(2,2)	$\langle 4,0\rangle$
(3,3)	$\langle 6,0\rangle$

8. (15 points) Let E be the solid bounded ~~above~~ below by the paraboloid $z = x^2 + y^2$ and ~~below~~ above by the half-cone $z = \sqrt{x^2+y^2}$. Set up but do not evaluate a triple integral to find the volume of this solid. Pick a coordinate system for which this triple integral is as simple as possible to evaluate.

$x^2+y^2 = \sqrt{x^2+y^2}$
when $x^2+y^2=1$
or $x=y=0$.

algebra:

$r^2 = r$

so $r^2 - r = 0$

so $r(r-1)=0$

so $\underline{\underline{r=0 \text{ or } r=1}}$

cylindrical * BEST *

$$\iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

rectangular

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} 1\,dz\,dy\,dx$$

spherical

$$= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{\cot\phi\csc\phi} \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Extra Credit: (5 points) Rewrite the triple integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx$$

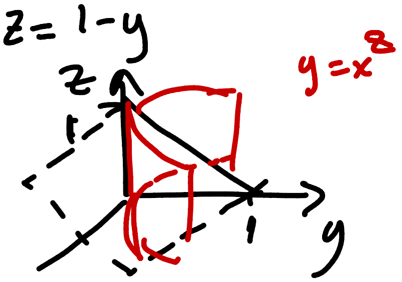
in the order $dx\,dy\,dz$.

$-1 \le x \le 1$

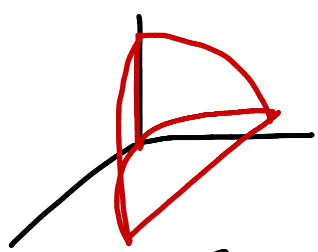
$\underline{x^2 \le y} \le 1$

$0 \le z \le 1-y$

$z = 1-y$



Solid:



$$= \int_0^1\int_0^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$