Your Name

Your Signature

Problem	Total Points	Score
1	10	
2	10	
3	12	
4	8	
5	14	
6	6	
7	10	
8	10	
9	10	
10	10	
Total	100	

- You have 1 hour.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.

1. (10 points) Determine whether or not the statements below are logically equivalent. Justify your answer. *Make sure you actually answer the question!*

$\boxed{(P \Rightarrow Q) \vee R}$ $\boxed{\sim((P \wedge \sim Q) \wedge \sim R)}$

P	Q	R	$P \Rightarrow Q$	$(P \Rightarrow Q) \vee R$	$P \wedge \sim Q \wedge \sim R$	$\sim((P \wedge \sim Q) \wedge \sim R)$
T	T	T	T	T	F	T
T	T	F	T	T	F	T
T	F	T	F	T	F	T
T	F	F	F	F	T	F
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	T	T	F	T

Because the two columns in pink are equal, the statement are logically equivalent.

2. (10 points) For each statement below, determine if the statement is true or false and **rigorously justify** your answer.

(a) $\forall X \in \mathscr{P}(\mathbb{Z}),\ X \subseteq \mathbb{Q}$

True. If $X \in P(\mathbb{Z})$, then $X \subseteq \mathbb{Z}$. Since $\mathbb{Z} \subseteq \mathbb{Q}$,

$X \subseteq \mathbb{Q}$.

(b) Given any real number x, there is some real number y such that $xy = 1$.

False.

Let $x=0$. Then $0 \cdot y = 0 = 1$ has no solution.

3. (12 points) Determine whether each statement below is True or False. Briefly justify your answer.

   (a) If $X = a, b, c$ and $Y = \{d, e\}$, then $X \cap Y = \{\emptyset\}$.

   False. $X \cap Y = \emptyset$ not $\{\emptyset\}$.

   (b) Let $A = \{1, 2, 3, 4\}$. Then $\{(2, 1)\} \in A \times A$.

   False. $(2,1) \in A \times A$. $\{(2,1)\} \subseteq A \times A$.

   (c) Let $A = \{1, 2, 3, 4\}$. Then $\{1, 2\} \in \mathscr{P}(A)$, where $\mathscr{P}(A)$ is the power set of A.

   True. $\{1,2\} \subseteq A$. So $\{1,2\} \in \mathscr{P}(A)$.

   (d) $|\ \{\ a,\ \{a, b\}, \{c, d, e, f\}\ \}\ | = 6$

   False. The set has $\underline{\underline{3}}$ elements, not 6.

4. (8 points) Write the following sets in set-builder notation.

   (a) $\{(1, 2),\ (2, 4),\ (3, 6),\ (4, 8),\ (5, 10), ...\} = \{(n, 2n) \mid n \in \mathbb{N}\}$

   $= \{(a, b) \mid a \in \mathbb{N} \text{ and } b = 2a\}$

   (b) $\{...,\ \frac{-1}{125},\ \frac{1}{25},\ \frac{-1}{5},\ 1,\ -5,\ 25,\ -125, ...\} = \{(-5)^n \mid n \in \mathbb{Z}\}$

5. (14 points) Write each of the following sets by listing its elements between braces or describing it with a familiar symbol or symbols.

(a) $\{n \in \mathbb{N} \mid (-1)^n = 1\} =$ $\{\ldots -4, -2, 0, 2, 4, \ldots\}$

(b) $\{X \in \mathscr{P}(\{a, b, c, d\}) \mid X \cup \{a, b\} = X\} =$ $\{\{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\}$

(c) $\mathscr{P}(\{a\}) \times \{1,2\} = \{(\emptyset,1), (\emptyset,2), (\{a\},1), (\{a\},2)\}$

$P(\{a\}) = \{\emptyset, \{a\}\}$

(d) $\bigcup_{n \in \mathbb{N}} [0, 1 + \frac{1}{n}] =$ $[0,2]$

(where $[a, b]$ denotes the standard interval notation)

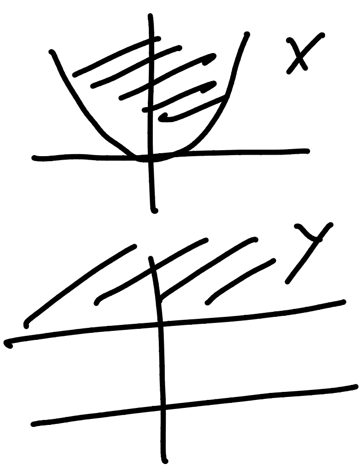
$[0,2], [0,\frac{3}{2}], [0, 1+\frac{1}{3}], \ldots$

(e) $\bigcap_{n \in \mathbb{N}} [0, 1 + \frac{1}{n}] =$ $[0,1]$

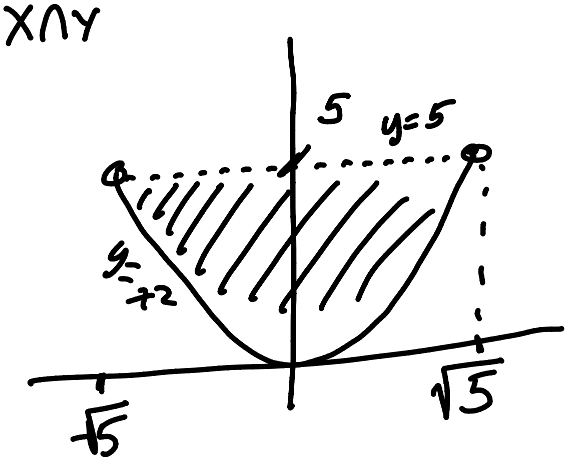
(where $[a, b]$ denotes the standard interval notation)

Label your graph.

6. (6 points) Let $X = \{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid y \geq 5\}$. Sketch $X \cap \overline{Y}$.



$X \cap \overline{Y}$



7. (10 points) Rewrite the following statements in the form "If P, then Q." Your answer should be a sentence in English.

   (a) A necessary condition for a function to be a polynomial is that the function is smooth.

necessary cond = conclusion.

if a function is polynomial, then the function is smooth.

   (b) The number x^2 is irrational only if x is irrational.

if x^2 is irrational, then x is irrational.

8. (10 points) Negate the statements below. Your answer must be a sentence in English and cannot contain the words "It is not the case that..."

   (a) For every $A \subseteq S$, there exists a subset $B \subseteq S$ such that $A \neq B$ and $A \subseteq B$.

There exists a subset $A \subseteq S$ such that for all subsets $B \subseteq S$, $A = B$ or $A \nsubseteq B$.

   (b) For every subset X of the natural numbers, if $|X|$ is infinite, then $|\overline{X}|$ is finite.

There exists a subset $X \subseteq \mathbb{N}$ such that $|X|$ is infinite and $|\overline{X}|$ is infinite.

9. (10 points) Let $S = \{A, B, C, D, E, F, G\}$, a set of **seven** symbols. For each of the counting problems below, you may leave your answer as a product. For example, all of the following would be acceptable forms of an answer: $8 \cdot 7 \cdot 6^3$ or $5 \cdot P(30, 5)$ or $\binom{15}{4}$.

   (a) List two *distinct* 4-permutations of S and determine the number of 4-permutations of S.

   ABCD
   ABDC

   $P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = \frac{7!}{3!}$

   (b) How many lists of length four can be made from S such that the list contains at least one repeated letter?

   $7! - P(7,4) = 7! - \frac{7!}{3!}$

   (c) How many lists of length 20 can be made from S such that the letter A appears exactly twice?

   $\binom{20}{2} \cdot 6^{18}$

10. (10 points) For the following counting problems, your answer should be an integer.

   (a) Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Determine the number of 4-element subsets of S.

   $\binom{8}{4} = \frac{8!}{4!4!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 3 \cdot 5 = 105$

   (b) Let $A = \{a, b, c\}$. Find $|\mathscr{P}(A)|$.

   $2^3 = 8$

   (c) Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f\}$. Find $|A \times B|$.

   $3 \cdot 5 = 15$