Your Name

Your Signature

Problem	Total Points	Score
1	15	
2	10	
3	15	
4	15	
5	20	
6	15	
7	10	
Total	100	

- You have 1 hour.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.

1. (a) (5 points) Complete the following *formal* definition:

   Given integers a and b, we say a *divides* b if

   there exists an integer n such that $b = an$.

   (b) (10 points) Suppose a, b, and c are integers. Use a direct proof to prove that if $a \mid b$ and $a \mid (b^2 + c)$, then $a \mid c$.

   Proof: Suppose a, b, and c are integers, $a \mid b$, and $a \mid (b^2 + c)$. Since $a \mid b$ and $a \mid (b^2 + c)$, we know (by definition) that there exist integers m and n such that $am = b$ and $an = b^2 + c$. Thus,

$$\begin{aligned} c &= an - b^2 && \text{rewriting the second equation} \\ &= an - a^2m^2 && \text{substituting in for } b \\ &= a(n - am^2). \end{aligned}$$

   Since, $n = am^2 \in \mathbb{Z}$, we have shown that $a \mid c$. $\square$

2. (10 points) Use the method of Proof by Contrapositive to prove the proposition below.

   Proposition: Suppose $a, b, c \in \mathbb{Z}$. Prove that if $a \nmid bc$, then $a \nmid b$ and $a \nmid c$.

   Proof: (by contrapositive) Suppose $a, b, c \in \mathbb{Z}$. We will prove that if $a \mid b$ or $a \mid c$, then $a \mid bc$.
   Suppose $a \mid b$. Then, by definition, there exists $n \in \mathbb{Z}$ such that $b = an$. Thus, $bc = anc = a(nc)$. Since c and n are both integers, cn is an integer. Thus, $bc = a(nc)$ demonstrates that $a \mid bc$.
   Suppose $a \mid c$. Then, since multiplication is commutative, the same argument shows that $a \mid bc$.

3. (a) (5 points) Complete the following *formal* definition:
   Given $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, we write $a \equiv b \mod n$ if
   $n \mid (a - b)$.

(b) (10 points) Suppose $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $a \equiv b \mod n$ and $a \equiv c \mod n$, then $2a \equiv b + c \mod n$.

Proof: (direct) Suppose Suppose $a, b, c \in \mathbb{Z}$, $n \in \mathbb{N}$, $a \equiv b \mod n$ and $a \equiv c \mod n$. By the definition of congruence, $n \mid (a - b)$ and $n \mid (a - c)$. From the definition of divides, it follows that $k_1 n = a - b$ and $k_2 n = a - c$, for some $k_1, k_2 \in \mathbb{Z}$. Now,

$$\begin{aligned} 2a - (b + c) &= (a - b) + (a - c) \\ &= k_1 n + k_2 n \\ &= (k_1 + k_2)n, \end{aligned}$$

where we know that $k_1 + k_2 \in \mathbb{Z}$. Thus, $n \mid 2a - (b + c)$. Thus, $2a \equiv (b + c) \mod n$.

4. (10 points) Any method may be used to prove the proposition below but you must state explicitly the method you are using.
Proposition: Suppose $x, y \in \mathbb{R}$. If $xy - x^2 + x^3 \geq x^2y^3 + 4$, then $x \geq 0$ or $y \leq 0$.

Proof: (by contradiction) Suppose $x, y \in \mathbb{R}$, $xy - x^2 + x^3 \geq x^2y^3 + 4$ and it is not the case that $x \geq 0$ or $y \leq 0$. Thus, we know $x < 0$ and $y > 0$. Now all the terms on the left side of the inequality, namely xy, $-x^2$ and x^3 are negative. In addition, all the terms on the right had side, namely x^2y^3 and 4, are positive. Now we have contradicted the assumption that $xy - x^2 + x^3 \geq x^2y^3 + 4$ since the left side is negative and the right side is positive.
Thus, we conclude that if $xy - x^2 + x^3 \geq x^2y^3 + 4$, then $x \geq 0$ or $y \leq 0$.

5. (a) (5 points) Complete the following *formal* definition:
The integer n is *even* if there exists $k \in \mathbb{Z}$ such that $n = 2k$.

(b) (15 points) Let a be an integer. Prove that a is even if and only if $a^2 + 2a + 9$ is odd.

Proof: Let a be an integer.
$\Rightarrow$: First we will show that if a is even, then $a^2 + 2a + 9$ is odd.
We will proceed using a direct proof. Assume a is even. Then by definition there exists an integer k such that $a = 2k$. Thus, $a^2 + 2a + 9 = (2k)^2 + 2(2k) + 9 = 2(2k^2 + 2k + 4) + 1$, where $2k^2 + 2k + 4 \in \mathbb{Z}$. Thus, we have shown that $a^2 + 2a + 9$ is odd.

$\Leftarrow$: Second we will show that if $a^2 + 2a + 9$ is odd, then a is even.
We will proceed by contrapositive. Thus, we will show that if a is odd, then $a^2 + 2a + 9$ is even.
Suppose a is odd. Then by definition there exists an integer k such that $a = 2k + 1$. Thus, $a^2 + 2a + 9 = (2k+1)^2 + 2(2k+1) + 9 = 2(k^2 + 4k + 6)$, where $k^2 + 4k + 6 \in \mathbb{Z}$. Thus, we have shown that $a^2 + 2a + 9$ is even.

Thus, a is even if and only if $a^2 + 2a + 9$ is odd.

6. (a) (5 points) Complete the following *formal* definition:
   Given sets A and B, we write $A \subseteq B$ if for every $a \in A$, we know $a \in B$.

   (b) (10 points) Suppose A, B and C are nonempty sets. Prove that if $A \times B \subseteq B \times C$, then $A \subseteq C$.

   Proof: Suppose A, B and C are nonempty sets and that $A \times B \subseteq B \times C$. Let $a \in A$ and let $b \in B$. Then $(a, b) \in A \times B$, by the definition of a cartesian product. Since $A \times B \subseteq B \times C$, we know $(a, b) \in B \times C$. Thus, by the definition of $B \times C$, it follows that $a \in B$. Thus, we have shown that $A \subseteq B$.
   Let $a \in A$. Since $A \subseteq B$, we know that $a \in B$. Thus $(a, a) \in A \times B$, by the definition of $A \times B$. Since $A \times B \subseteq B \times C$, we know $(a, a) \in B \times C$. Thus, by the definition of $B \times C$, it follows that $a \in C$. Thus, we have shown that $A \subseteq C$.

7. (10 points) Prove that there exists a set X such that $\mathbb{N} \in X$ and $\mathbb{R} \in \mathscr{P}(X)$.

   Proof: Let $X = \mathbb{R} \cup \{\mathbb{N}\}$. (So, X is the set of all real numbers to which I have added a single additional element called the set of all natural numbers.) Thus, $\mathbb{N} \in X$ since this element is added explicitly. In addition , $\mathbb{R} \in \mathscr{P}(X)$ since for every $r \in \mathbb{R}$, $r \in X$.