Your Name: solutions

Your Signature:

1. (20 points) Disprove the following two statements.

   (a) For all sets $A$, $B$ and $C$, if $A \not\subseteq B$ and $B \not\subseteq C$, then $A \not\subseteq C$.

   We must prove the negation of the statement: There exist sets $A$, $B$ and $C$ such that $A \not\subseteq B$, $B \not\subseteq C$, and $A \subseteq C$. Thus, a simple counter-example is sufficient.

   Example: Let $A = \{1\}, B = \{2,3\}$ and $C = \{1,2\}$. Observe that $A \not\subseteq B$, $B \not\subseteq C$, and $A \subseteq C$.

   (b) There exists a natural number $n$ such that $3 \mid n$ and $3 \mid (n+1)$.

   We must prove the negation of the statement: For every natural number $n$ either $3 \nmid n$ or $3 \nmid (n+1)$.

   (direct) Let $n$ be an arbitrary natural number. If $3 \nmid n$, then the statement holds. If $3 \mid n$, then there exists an integer $k$ such that $3k = n$. Thus, $n+1 = 3k+1$. Now $3 \nmid (3k+1)$ since $3 \nmid 1$.

   (by contradiction) Suppose $n$ is a natural number such that $3 \mid n$ and $3 \mid (n+1)$. Thus, there exist integers $k$ and $\ell$ such that $3k = n$ and $3\ell = n+1$. Thus, we have the contradiction that

$$1 = (n+1) - n = 3\ell - 3k = 3(\ell - k)$$

   implies $3 \nmid 1$. Thus, no such $n$ can exist.

2. (10 points) Prove that for all integers $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

**Proof:** (by induction on $n$)
Base Step: Let $n = 2$. Observe that $\left(1-\frac{1}{2^2}\right)=\frac{3}{4}=\frac{2+1}{2\cdot 2}$. Thus, the proposition holds for $n = 2$.

Inductive Step: Let $k \in \mathbb{N}$ such that $k \geq 2$. Suppose that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}.$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2}.$$

Observe

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right) &= \left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\right]\left(1-\frac{1}{(k+1)^2}\right)\\ &= \left[\frac{k+1}{2k}\right]\left(1-\frac{1}{(k+1)^2}\right)\\ &= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\ &= \left(\frac{1}{2k}\right)\left(\frac{k(k+2)}{(k+1)}\right)\\ &= \frac{k+2}{2k+2},\end{aligned}$$

where the inductive hypothesis is used in line 2 above and the remainder is algebra. Thus, we have shown that if the proposition holds for index $k$, then it holds for index $k+1$.

Thus, we have shown by induction that for all integers $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

3. (10 points) Suppose $A$, $B$ and $C$ are sets. Prove that $A \subseteq B$ if and only if $A - B = \emptyset$. (Hint: You may not want to use the method of direct proof here.)

**Proof:** (Option 1: by contrapositive) Let $A$, $B$ and $C$ be sets. Observe that the statement $\boxed{A \subseteq B \text{ if and only if } A - B = \emptyset}$ is equivalent to the statement $\boxed{A \not\subseteq B \text{ if and only if } A - B \neq \emptyset.}$ We will prove the second equivalent statement.

$\Rightarrow$: Suppose $A \not\subseteq B$. Thus, by definition, there exists an element $a \in A$ such that $a \notin B$. Thus, $a \in A - B$, and so $A - B \neq \emptyset$.

$\Leftarrow$: Suppose $A - B \neq \emptyset$. Then, there exists an element $a \in A - B$. Thus, $a \in A$ and $a \notin B$. Thus, $A \not\subseteq B$.

**Proof:** (Option 2: by contradiction) Let $A$, $B$ and $C$ be sets.
$\Rightarrow$: Suppose $A \subseteq B$ and $A - B \neq \emptyset$. Since $A - B \neq \emptyset$, there exists an element, say $a$, in $A - B$. So, $a \in A$ and $a \notin B$. But this implies that $A \not\subseteq B$, which contradicts the assumption that $A \subseteq B$.

$\Leftarrow$: Suppose $A - B = \emptyset$ and $A \not\subseteq B$. Since $A \not\subseteq B$, there must exist some $a \in A$, such that $a \notin B$. But if such an element $a$ exists, then $a \in A - B$ which contradicts the assumption that $A - B = \emptyset$.

**Proof:** (Option 3: direct) Let $A$, $B$ and $C$ be sets.

$\Rightarrow$: Suppose $A \subseteq B$. Thus, by the definition of subset, if $a \in A$, then $a \in B$. Thus, there does not exist any element $x$ such that $x \in A$ and $x \notin B$. Thus, there exists no element $x$ such that $x \in A - B$. Thus, $A - B = \emptyset$, which is what we needed to show.

$\Leftarrow$: Suppose $A - B = \emptyset$. Since the set $A - B$ contains no elements, by the definition of set difference, it follows that there does not exist a single element $x$ such that $x \in A$ and $x \not{i}nB$. Hence, for every $a \in A$, it must be that $a \in B$. Thus, by definition of subset, $A \subseteq B$, which is what we wanted to show.

4. (10 points) Use induction to prove that for every integer $n$ such that $n \geq 2$, $5^n + 9 < 6^n$.

**Proof:** (by induction on $n$)
Base Step: Let $n = 2$. Observe that $5^2 + 9 = 34 < 36 = 6^2$. Thus, the proposition holds for $n = 2$.

Inductive Step: Let $k$ be an integer such that $k \geq 2$. Suppose that $5^k + 9 < 6^k$. We want to show that $5^{k+1} + 9 < 6^{k+1}$. First note that if $5^k + 9 < 6^k$, then $5^k < 6^k - 9$. Observe

$$\begin{aligned}
5^{k+1} + 9 &= 5(5^k) + 9 \\
&< 5(6^k - 9) + 9 && \text{by the inductive hypothesis} \\
&= 5 \cdot 6^k - 45 + 9 \\
&< 5 \cdot 6^k + 9 && \text{because } -45 < 0 \\
&< 6 \cdot 6^k + 9 && \text{because } 5 < 6 \\
&= 6^{k+1} + 9,
\end{aligned}$$

which is what we wanted to show. Thus, if the proposition holds for $k$, it holds for $k + 1$. Thus, by induction, the proposition is true for all integers $n \geq 2$.

5. (10 points) Prove that for all sets $A$ and $B$, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. (Note $\mathcal{P}(A)$ is the power set of the set $A$.)

   **Proof:** Let $A$ and $B$ be sets. Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Thus, $X \subseteq A$ or $X \subseteq B$. If $X \subseteq A$, then $X \subseteq A \cup B$. Thus, $X \in \mathcal{P}(A \cup B)$. If $X \subseteq B$, then $X \subseteq A \cup B$. Thus, $X \in \mathcal{P}(A \cup B)$. Thus, we have shown that if $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$, then $X \in \mathcal{P}(A \cup B)$. Thus, it follows that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(5 points extra credit) Suppose $a, b \in \mathbb{N}$. Then $a = lcm(a, b)$ if and only if $b \mid a$.

**Proof:** Suppose $a, b \in \mathbb{N}$.

$\Rightarrow$: Suppose $a = lcm(a, b)$. Then $a = bn$ for some integer $n$. Hence $b \mid a$.

$\Leftarrow$: Suppose $b \mid a$. Then $a = bn$ for some integer $n$ and $a = a \cdot 1$, thus the number $a$ is a common multiple of $a$ and $b$. So $a \geq lcm(a, b)$. On the other hand, $lcm(a, b) \geq a$ since any multiple of $a$ is at least $1 \cdot a$.

Since $a \geq lcm(a, b)$ and $a \leq lcm(a, b)$, it follows that $a = lcm(a, b)$.