Math 265
April 3, 2022

Ch 11: Relations

1. State the definitions of

   (a) a relation R on a set A.

   (b) a reflexive relation

   (c) a symmetric relation

   (d) a transitive relation

   (e) an equivalence relation

2. Let $n \in \mathbb{N}$. Prove that the relation R on $\mathbb{Z}$ defined as $a\,R\,b$ if $a \equiv b \bmod n$ is transitive.

Let $a, b, c \in \mathbb{Z}$ s.t. $a \equiv b \bmod n$ and $b \equiv c \bmod n$.
We n.t.s. $a \equiv c \bmod n$.
Since $a \equiv b \bmod n$ and $b \equiv c \bmod n$, by definitin,
$\exists$ integers k, ℓ so that

$$a - b = kn \quad \text{and} \quad b - c = \ell n.$$

So $a - c = (a-b) - (b-c) = kn - \ell n = (k-\ell)n$.
Since $k - \ell \in \mathbb{Z}$, it follows that $a \equiv c \bmod n$.

3. For each relation below, determine whether it is reflexive, symmetric, or transitive.

(a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 \leq 4\}$

reflexive? No. $(5,5) \notin \mathbb{R} \times \mathbb{R}$ because $5^2 + 5^2 \leq 4$

symmetric? Yes. If $(x,y) \in R$, then $x^2 + y^2 \leq 4$.

So $y^2 + x^2 \leq 4$. So $(y,x) \in R$.

transitive? No. Observe that $(2,0), (0,\sqrt{3}) \in R$ since $2^2 + 0^2 = 4$ and $0^2 + (\sqrt{3})^2 = 3$ are both at most 4.

But $(2,\sqrt{3}) \notin R$, since $2^2 + (\sqrt{3})^2 = 4 + 3 = 7$.

(b) R is a relation on $\mathcal{P}(\mathbb{N})$ such that ARB if $|A - B| \leq 2$.

reflexive? Yes. $\forall A \subseteq \mathbb{N}$, $|A - A| = 0 \leq 2$.

Symmetric? No. Let $A = \{1\}$, $B = \{2,3,4\}$.

Then $|A - B| = |A| = 1$ but $|B - A| = |B| = 3$.

transitive? No. Let $A = \{1,2,3\}$, $B = \{2,3,4\}$ and $C = \{4\}$.

Then $|A - B| = |\{1\}| = 1$ and $|B - C| = |\{2,3\}| = 2$.

So ARB and BRC.

But $|A - C| = |\{1,2,3\}| = 3$. So $A \not R C$.

(c) R is a relation on $\mathcal{P}(\mathbb{N})$ such that ARB if $A - \{1,2\} = B - \{1,2\}$.

reflexive? Yes. $\forall A \subseteq \mathbb{N}$, $A - \{1,2\} = A - \{1,2\}$.

symmetric? Yes. $\forall A, B \subseteq \mathbb{N}$, if $A - \{1,2\} = B - \{1,2\}$,
then $B - \{1,2\} = A - \{1,2\}$.
This if ARB, then BRA.

transitive? Yes. Let $A, B, C \in \mathcal{P}(\mathbb{N})$ s.t. ARB and BRC.
Then $A - \{1,2\} = B - \{1,2\} = C - \{1,2\}$. So $A - \{1,2\} = C - \{1,2\}$
So ARC.

(d) R is a relation on $\mathbb{Z}$ defined as $(m, n) \in R$ if $3m - 5n$ is even.

reflexive: Yes. $\forall m \in \mathbb{Z}$, $3m - 5m = -2m$ which is even

symmetric Yes Let $m, n \in \mathbb{Z}$ s.t. $(m,n) \in R$.
Then $3m - 5n$ is even. So $3m - 5n = 2k$.
Now $3n - 5m = 3n - 5m + 8m - 8m + 8n - 8n$
$= 3m - 5n + 8(n - m)$
$= 2k + 8(n-m)$, which is even.

transitive: Yes. Let $m, n, p \in \mathbb{Z}$ s.t. $(m,n), (n,p) \in R$.
Thus, $3m - 5n$ and $3n - 5p$ is even.
Thus, $3m - 5n + 3n - 5p$ is even. Thus
$3m - 5n + 3n - 5p = 3m - 5p - 2n = 2k$, for $k \in \mathbb{Z}$.
Thus, $3m - 5p = 2(k + n)$. So $3m - 5p$ is even.
Thus $(m, p) \in R$.