Math 265
March 3, 2022

Ch 8 & Ch 9

1. Recall that we proved the following result in class:

   Let $a, b \in \mathbb{N}$. If $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.

2. Let p and q be distinct prime numbers and let c be an integer. Prove that if $p \mid qc$ then $p \mid c$.

   Observe that since p and q are distinct primes, $\gcd(p, q) = 1$. Since $\gcd(p, q) = 1$ and, by hypothesis, $p \mid qc$ where $c \in \mathbb{Z}$, we can apply the result in item 1 above to conclude $p \mid c$.

3. Prove that the previous statement is false without the hypothesis that p and q are distinct prime numbers.

   Let p be an odd prime and let $q = 2p$ and $c = 2$. Then $qc = 4p$ and $p \mid qc$ but $p \nmid c$.

4. Prove one of DeMorgan's Laws:
   Let A and B be sets with universe U. Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

   proof: Let A and B be sets with universe U.

   First we will show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

   Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$. Thus, $x \notin A$ and $x \notin B$. Thus, $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \in \overline{A} \cap \overline{B}$. Thus, we have shown that if $\overline{A} \cap \overline{B}$, then $x \in \overline{A} \cap \overline{B}$. Thus, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

   Next we will show that $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

   Let $x \in \overline{A} \cap \overline{B}$. Thus, $x \in \overline{A}$ and $x \in \overline{B}$. Thus, $x \notin A$ and $x \notin B$. Thus, $x \notin A \cup B$. Thus, $x \in \overline{A \cup B}$. Thus, we have shown that if $x \in \overline{A} \cap \overline{B}$, then $\overline{A} \cap \overline{B}$. Thus, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

   Because $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, it follows that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

5. We said in class that we can think of the statement $\boxed{A \subseteq B}$ as equivalent to the statement $\boxed{\text{If } a \in A, \text{ then } a \in B.}$

   Write the contrapositive and the negation of the boxed statements.

contrapostive: If $a \notin B$, then $a \notin A$.

negation: There is an element x such that $x \in A$ and $x \notin B$.

6. Prove the proposition below using the contrapositive and the negation.

Proposition: $\{n \in \mathbb{Z} : 4 \mid n\} \subseteq \{n \in \mathbb{Z} : 2 \mid n\}$.

Proof by contrapositive
Let $A = \{n \in \mathbb{Z} : 4 \mid n\}$ and let $B = \{n \in \mathbb{Z} : 2 \mid n\}$. Suppose $x \notin B$. Then, by the definition of B, x is not an integer or x is not divisible by 2. If x is not an integer, then $x \notin A$. On the other hand, if x is an integer but is not divisible by 2, then x is an odd integer. If x is odd, then x is not divisible by 4 since all integers divisible by 4 are even. Since x is not divisible by 4, $x \notin A$.

Thus, in both cases, if $x \notin B$, it follows that $x \notin A$. Thus, we have shown that $A \subseteq B$.

Proof by contradiction
Let $A = \{n \in \mathbb{Z} : 4 \mid n\}$ and let $B = \{n \in \mathbb{Z} : 2 \mid n\}$. Suppose $x \in A$ and $x \notin B$. Since $x \in A$, x is divisible by 4 and there exists an integer k such that $4k = x$. Thus, $x = 2(2k)$ where $2k \in \mathbb{Z}$ demonstrating that x is even.

Since $x \notin B$, it follows that $2 \nmid x$. Thus, x is odd.

Now we have the contradiction that x is both even and odd. Thus it cannot be the case that $x \in A$ and $x \notin B$. Thus, we have shown that $A \subseteq B$.