Math 265
March 18, 2022

Wednesday 16 March

1. Let A, B, and C be sets. If $A \times C = B \times C$, then $A = B$.

   Proof: Let A, B, and C be sets and assume $A \times C = B \times C$.

   Let $a \in A$ and let $c \in C$. Thus, by definition $(a, c) \in A \times C$. Since by assumption $A \times C = B \times C$ and $(a, c) \in A \times C$, it follows that $(a, c) \in B \times C$. Since $(a, c) \in B \times C$, it follows that $a \in B$. Thus, we have shown that if $a \in A$, then $a \in B$. Thus, $A \subseteq B$.

   If we switch A and B in the previous paragraph, we prove that $B \subseteq A$.

   Since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$.

2. If $x, y \in \mathbb{R}$ and $x^3 < y^3$, then $x < y$.

   Proof: Let $x, y \in \mathbb{R}$ and $x^3 < y^3$. Proceed by contradiction and assume that $x \geq y$. Using algebra, we obtain:
   $$0 < y^3 - x^3 = (y - x)(x^2 + xy + y^2). \tag{1}$$
   Since $x \geq y$, we know that $y - x \leq 0$. If $y - x = 0$, then expression 1 gives:
   $$0 < y^3 - x^3 = (y - x)(x^2 + xy + y^2) = 0 \cdot (x^2 + xy + y^2) = 0, \tag{2}$$
   a contradiction. On the other hand, if $x - y < 0$, then expression 1 implies that $x^2 + xy + y^2 < 0$ in order for the product to be positive. Thus, $xy < 0$. So $y < 0$ and $x > 0$. But this implies $y^3 < 0$ and $x^3 > 0$. But this contradicts the assumption that $y^3 < x^3$.

   Since a contradiction is obtained in both cases, we can conclude that $x < y$.

3. For every $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

   Proof: (by induction)

   Base Case: If $n = 1$, then $\displaystyle\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$. Thus, the proposition holds for $n = 1$.

   Inductive Case: Let $k \in \mathbb{N}$. (So $k \geq 1$.) Suppose that $\displaystyle\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$. We must show that $\displaystyle\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.
   Observe

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= 1+2+3+\cdots+(k-1)+k+(k+1) && \text{expanding summation notation} \\
&= (1+2+3+\cdots+(k-1)+k)+(k+1) && \text{associativity of addition} \\
&= \left(\sum_{i=1}^{k} i\right)+(k+1) && \text{contracting summation notation} \\
&= \left(\frac{k(k+1)}{2}\right)+(k+1) && \text{inductive hypothesis} \\
&= \left(\frac{k(k+1)}{2}\right)+\frac{2(k+1)}{2} && \text{algebra} \\
&= \left(\frac{(k+2)(k+1)}{2}\right),
\end{aligned}
$$

which is what we wanted to show.

If follows by induction that for every $n \in \mathbb{N}$, $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.