Your Name

Solutions

Your Signature

Problem	Total Points	Score
1	15	
2	14	
3	14	
4	15	
5	10	
6	12	
7	10	
8	10	
Total	100	

- You have 1 hour.
- If you have a cell phone with you, it should be turned off and put away. (Not in your pocket)
- You may not use a calculator, book, notes or aids of any kind.
- In order to earn partial credit, you must show your work.

1. (15 points)

   (a) Complete the definition below.

   Given integers a and b and $n \in \mathbb{N}$, we say that a and b are congruent modulo n if

   $n \mid (a-b)$

   (b) **Use the definition and a direct proof** to prove the statement below. Do not use any previous results from the text or in homework.

   If $a \in \mathbb{Z}$ and $a \equiv 1 (\text{mod } 7)$, then $a^2 \equiv 1 (\text{mod } 7)$.

Pf:

- Supps $a \in \mathbb{Z}$ and $a \equiv 1 (\text{mod} 7)$.

Thus, $7 \mid a-1$ or, equivalently, $\exists\, n \in \mathbb{Z}$ s.t. $\boxed{7n = a-1.}$

Multiply both sides by $a+1$ to obtain:

$$7n(a+1) = (a-1)(a+1) = a^2 - 1.$$

~~Since $n(a+1) \in \mathbb{Z}$~~,

Observe that $n(a+1) \in \mathbb{Z}$, say $m = n(a+1)$

- Thus, $7m = a^2 - 1$ where $\boxed{m \in \mathbb{Z}.}$
- Thus, $7 \mid a^2 - 1$.
- Thus, $a^2 \equiv 1 (\text{mod} 7)$.

+7

2. (14 points)

   (a) List the elements in the set $\{x \in \mathbb{Z} : |3x| \leq 6\}$.

   $-2, -1, 0, 1, 2$

+3

   (b) List the elements in the set $\{X \subseteq \{a, b, c\} : a \notin X\}$.

   $\emptyset, \{b\}, \{c\}, \{b, c\}$

+4

   (c) Write the set $\{\cdots, \frac{-\pi}{4}, \frac{-\pi}{2}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \cdots\}$ in set-builder notation.

+4 $\{\frac{\pi}{4}k : k \in \mathbb{Z}\}$

   (d) Determine the cardinality of the set $\{\emptyset, \{\emptyset\}, \{1, 2\}, \{1, 2, 3\}\}$.

+3 4

$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \ldots$
$\{0,1\}, \{0,2\}, \ldots$
$\{0,1,2\}, \{0,1,3\}, \ldots$
$\{1,2,3,4\}, \{1,2,0,4\} \ldots$

3. (14 points) Let $A = \{0, 1, 2, 3, 4\}$ and $\mathcal{P}(A)$ denote the power set of A.

   (a) Determine $|\mathcal{P}(A)|$, the cardinality of $\mathcal{P}(A)$. $2^5 = 32$

4

   (b) List 3 distinct **elements** of $\mathcal{P}(A)$ such that each element has a different cardinality. Use correct notation.

5 $\emptyset, \{0\}, \{0, 1\}$

   (c) List 3 distinct **subsets** of $\mathcal{P}(A)$ such that each subset has different cardinality. Use correct notation.

5 $\{\emptyset\}, \{\emptyset, \{0\}\}, \{\emptyset, \{0\}, \{0, 1\}\}$

4. (15 points) Let $A = \{0, 1, 2\}, B = \{1, 2, 3, 4\}$ and define the universal set $U = \{0, 1, 2, 3, \cdots, 9\}$. Find:

(a) $A \cup B$ $= \{0,1,2,3,4\}$

+3

(b) $\overline{A \cup B}$ $= \{5,6,7,8,9\}$

+3

(c) $|A \times B|$ $= 3 \times 4 = 12$

+3

(d) $(A \times A) \cap (B \times B)$ $= \{(x,y) : x \in \{1,2\}, y \in \{1,2\}\}$

+3 $= \{(1,1), (1,2), (2,1), (2,2)\}$

(e) $(A \times A) - (A \times B)$

+3 $= \{(x,y) : x \in \{0,1,2\}, y = 0\} = \{(0,0), (1,0), (2,0)\}$

5. (10 points) Complete the truth table for the statement $P \Leftrightarrow (Q \vee \sim R)$.

P	Q	R	$\sim R$	$Q \vee \sim R$	$P \Leftrightarrow (Q \vee \sim R)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	F
F	F	T	F	F	T
F	F	F	T	T	F

6. (12 points) Negate the two statements below. Your answer should be a complete sentence in English. (You are not asked to determine the truth value of these statements.)

(a) There exists a real number r such that $r > 1$ and $r^2 < 1.001$. $\exists r, r>1 \wedge r^2<1.001$.

For every real number r, $r \leq 1$ or $r^2 \geq 1.001$.

-1

(b) If $a \in X$, then $a \notin Y - X$. $\forall a \in X, a \notin Y-X$.

There is an $a \in X$ such that $a \in Y-X$.

7. (10 points) Prove the statement below with a contrapositive proof.

Let $x, y \in \mathbb{Z}$. If $3x - 5y$ is odd, then x and y do not have the same parity.

~~Pf~~ Contrapositive:

let $x,y \in \mathbb{Z}$. If x and y have the same parity, then $3x-5y$ is even.

Pf: Let $x,y \in \mathbb{Z}$ and suppose x and y have the same parity.

Case 1: Suppse x and y are even

Then $x=2m$ and $y=2n$ for $m,n \in \mathbb{Z}$. Thus

$$3x-5y = 3(2m)-5(2n) = 2(3m-5n),$$ where

$3m-5n \in \mathbb{Z}$.

Thus, $3x-5y$ is even if x and y are even.

Case 2 Sppse x and y are odd.

Then $x=2m+1$ and $y=2n+1$ for $m,n \in \mathbb{Z}$. Thus,

$$3x-5y = 3(2n+1)-5(2n+1) = 2(-2n-1)+1,$$

where $-2n-1 \in \mathbb{Z}$.

Thus, $3x-5y$ is odd if x and y are odd.

8. (10 points) Prove the statement below using a proof by contradiction.

Let $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a is even or b is even.

Negation: $4 \mid (a^2+b^2)$ and a and b are both odd.

Pf: (by contradiction)

Suppose $4 \mid (a^2+b^2)$ and a and b are both odd.

By definition of odd, $\exists\, m, n \in \mathbb{Z}$ such that

$a = 2n+1$ and $b = 2m+1$.

So $a^2+b^2 = 4n^2+4n+1+4m^2+4m+1 = 4(n^2+n+m^2+m)+2$,

where $n^2+n+m^2+m \in \mathbb{Z}$, say k.

On the other hand, since $4 \mid a^2+b^2$, $\exists\, l \in \mathbb{Z}$ such that

$4l = a^2+b^2$.

So, $4l = a^2+b^2 = 4k+2$. Thus, $4(l-k) = 2$.

~~So~~ So $4 \mid 2$ which is a contradiction

since $4 \nmid 2$.

Thus, we have shown that if $4 \mid a^2+b$, then

either a is even or b is even.