Your Name

Problem	Total Points	Score
1	16	
2	10	
3	15	
4	16	
5	15	
6	12	
7	16	
Total	100	

- This test is closed book.
- A student may bring one 8×11 sheet of paper with writing on the front.
- No calculator is needed since all answers can be left in "choose" form. (So $\boxed{6^8\binom{10}{4}}$ is an acceptable answer.)
- In order to receive full credit, you must **show your work.**
- Raise your hand if you have a question.

1. (16 points) Let $S(n, k)$ denote the Stirling numbers of the second kind.

   (a) Use complete enumeration of appropriate set partitions to determine $S(4, 2)$.

   (b) Give a combinatorial justification that $S(n, 2) = 2^{n-1} - 1$. (That is, we were given a formula for $S(n, k)$. You cannot use this formula. Essentially, you are asked to show that this formula is correct when $k = 2$.)

   (c) How many different functions with domain $[n]$ and **codomain** $[k]$ are possible?

   (d) How many different functions with domain $[n]$ and **range** $[k]$ are possible?

2. (10 points) Use inclusion-exclusion to determine how many integers in [100] are not divisible by 4, 6, or 7. A calculation is sufficient. You do not need to simplify your answer.

3. (15 points) Let $P(n, k)$ count the number of integer partitions of n into k parts.

   (a) Use complete enumeration to determine $P(6, 3)$.

   (b) Give a combinatorial proof of the identity below:

   If $n \geq 1$ and $k \geq 1$, then $P(n, k) = P(n-1, k-1) + P(n-k, k)$.

4. (16 points)

(a) Write an ordinary generating function to count the number of ways to distribute 20 identical pieces of candy to two adults and three children. Assume the adults receive at most one piece of candy and the children receive at least one piece of candy. Identify what coefficient you need.

(b) Let $g(x) = (1 + x^2 + x^4)(x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)^2$.

i. Write a concise form of $g(x)$.

ii. Find $[\![g(x)]\!]_{x^6}$. Simplify your answer.

iii. Given an example of a problem for which $[\![g(x)]\!]_{x^k}$ is an answer.

5. (15 points) Prove that $R(K_{1,3}, K_3) = 7$.

   (a) Draw and label the graphs $K_{1,3}$ and K_3.

   (b) Demonstrate that $R(K_{1,3}, K_3) > 6$. (Note that an example is not sufficient. You must explain how that example implies the lower bound.)

   (c) Demonstrate that $R(K_{1,3}, K_3) = 7$.

6. (12 points) Let $\mathcal{V} = [n]$ for an integer $n \geq 3$ and let k be an integer such that $2 \leq k \leq n - 1$.

   (a) Let $\mathcal{B}$ be the set of all k-subsets of $\mathcal{V}$. Does $(\mathcal{V}, \mathcal{B})$ form a balanced incomplete block design? Prove your answer is correct.

   (b) Let $B = \{1, 2, 3\} \subseteq \mathcal{V}$ and assume $n \geq 5$. Can B be the base block of a cyclic design? Prove your answer is correct.

7. (16 points)

(a) A store has 20 varieties of doughnuts. In how many ways can you fill an order for a dozen doughnuts assuming there are no restrictions on the types of doughnuts. (That is, you may choose several of the same variety. Assume the store has a least 12 of each variety.)

(b) A store has 20 varieties of doughnuts. In how many ways can you pick a different variety of doughnut for each for your 7 best friends.

(c) A nice teacher brings a box of 12 **identical** glazed doughnuts for her class of 8 (nonidentical) students. In how many ways can the doughnuts be given to the students assuming each student gets at least one doughnut but at most two doughnuts. (Assume the doughnuts are not divided.)

(d) A nice teacher brings a box of 12 **different** doughnuts for her class of 8 (nonidentical) students. In how many ways can the doughnuts be given to the students assuming each student gets at least one but there is no other restriction. (So some student could get 5 doughnuts. Assume that the order in which a student gets his/her doughnuts does not matter.)