Your Name

Problem	Total Points	Score
1	24	
2	12	
3	18	
4	15	
5	16	
6	15	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- No calculator is needed since all answers can be left in "choose" form. (So $\boxed{6^8\binom{10}{4}}$ is an acceptable answer.)
- In order to receive full credit, you must **show your work.**
- Raise your hand if you have a question.

1. (24 points) Let $S = \{a, b, c, d, e, f, g, h, i, j\}$ be a set of 10 letters. (Answers without work are acceptable.)

   (a) How many permutations of S have all of the vowels before all of the consonants?

   $3! \cdot 7!$

   (b) How many 6-character passwords are possible using only letters from S?

   10^6

   (c) How many 6-character passwords have at least one repeated letter?

   $10^6 - (10)_6 = 10^6 - [10 \cdot 9 \cdot 8 \cdot 7 \cdot 6]$ Counting the complement

   (d) How many 6-character passwords have all characters in alphabetical order? (So the passwords **bcfhij** and **acdehi** would be counted but password **bcaefg** would not, since the letter a is out of order.)

   $\binom{10}{6}$ Once the letters have been chosen, the alphabet determines the order.

2. (12 points) (Answers with no work are OK.)

   (a) How many functions $f : [n] \to [k]$ are onto?

   $S(n, k) \cdot k!$

   (b) How large does n need to be to guarantee that, for every function $f : [n] \to [20]$, there exists some $b \in [20]$ such that $|f^{-1}(b)| \geq 3$?

   $n \geq 61$

3. (18 points) There are 100 pieces of candy and 35 children. Find the number of ways to distribute the candy to the children in each of the following situations.

   (a) The pieces of candy are indistinguishable and each child gets at least one piece. (The children are considered to be different from each other.)

   - Give each child a piece, leaving 65 pieces.

   - $\left(\!\!\binom{35}{65}\!\!\right) = \binom{35-1+65}{65} = \binom{94}{65}$

   (b) The pieces of candy are all different and each child gets exactly one piece of candy. (So some candy is left over.)

   $(100)_{35}$

   (c) The pieces of candy are all different but you distribute them among 35 identical paper bags assuming no bag is left empty.

   $S(100,35)$

4. (15 points) Give a **combinatorial** proof of the identity below.

For any positive integers m and n, $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$

Let S be a set of m men and n women.
The number of k-person commitees that can be formed from S is $\binom{|S|}{k} = \binom{m+n}{k}$.

Alternatively we could count the commitees by conditioning on the number of men on the committee. If there are j men, then $\binom{m}{j}$ is the number of ways to select them. Then there are $\binom{n}{k-j}$ ways to select the remaining k-j committee members from the n women. Since the number of men on the committee can range from none (j=0) to all (j=m), the number of k-person commites from S is also:

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}.$$

5. (16 points) The following question concerns the proposition below:

Proposition: If $n \geq 1$, then $S(n, 2) = 2^{n-1} - 1$.

(a) List all 2-partitions of [4] and show that the proposition holds for $n = 4$.

1, 234   12, 34
2, 134   13, 24
3, 124   14, 23
4, 123

$S(4,2) = 7$ and

$2^{4-1} - 1 = 2^3 - 1 = 7$

So $S(n,2) = 2^{n-1} - 1$ in this case.

(b) Give a **bijective** proof of the proposition.

Let A be the set of all 2-partitions of [n] and let B be the set of all nonempty subsets of [n-1].

Let $a \in A$. Then a has two blocks: one containing n and one that doesn't. Map partition a to the subset of [n-1] corresponding to the elements in the block NOT containing n. Clearly the image of a is a nonempty set (since blocks are nonempty) and it's a subset of [n-1] (since n is excluded.)

One-to-one: Let a and a′ be mapped to the same subset of [n-1], call it B. Then both a and a′ have B as one block and $\overline{B} \cup \{n\}$ as their other block. Thus a and a′ are the same partition.

onto: Let $B \subseteq [n-1]$ where $B \neq \emptyset$.

Construct $a \in A$ as follows: one block is B and the second is $\overline{B} \cup \{n\}$. Since a is a partition of [n] with two blocks, $a \in A$. By definition, a is mapped to B

6. (15 points)

   (a) By enumeration, determine $P(4,2)$, the number of 2-partitions of 4.

   3+1

   2+2

   (b) P(5,3)

   3+1+1
   2+2+1

   for $n \geq 4$ n-2

   (b) Determine a formula for $P(n,2)$, the number of 2-partitions of n, and explain why your answer is correct.

   $P(n, n-2) = 2$

   In order to form n-2 non zero parts, we require n-2 units (or 1's) from n. This leaves only 2 remaining. There are only two possibilities: put the last two together to form a part of size 3 or put them in separate parts to form two parts each of size 2.

Extra Credit: (5 points) Consider any 5 points in the xy-plane with integer coordinates. (That is, point $A(-3,18)$ had integer coordinates but point $B(2,4/3)$ does not.) Prove that there must exist two of the five points such that the midpoint of the line segment joining those two points also has integer coordinates.

For each point, there are two choices for the parity of each coordinate: (even, even), (even, odd), (odd, odd), (odd, even). There are only these four possibilities since $2^2 = 4$. With 5 points, two must have the same "parity type."

Now, the midpoint formula: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ gives integer coordinates for such a pair, since x_1+x_2 and y_1+y_2 must both be even.