Your Name

Problem	Total Points	Score
1	24	
2	12	
3	18	
4	15	
5	15	
6	16	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- No calculator is needed since all answers can be left in "choose" form. (So $\boxed{6^8\binom{10}{4}}$ is an acceptable answer.)
- In order to receive full credit, you must **show your work.**
- Raise your hand if you have a question.

1. (24 points) Let $S = \{a, b, c, d, e, f, g, h, i, j\}$ be a set of 10 letters. (Answers without work are acceptable.)

   (a) How many permutations of S have all of the vowels before all of the consonants?

   (b) How many 6-character passwords are possible using only letters from S?

   (c) How many 6-character passwords have at least one repeated letter?

   (d) How many 6-character passwords have all characters in alphabetical order? (So the passwords **bcfhij** and **acdehi** would be counted but password **bcaefg** would not, since the letter a is out of order. Indeed the letters in the password must all be distinct.)

2. (12 points)

   (a) How many functions $f : [n] \to [k]$ are onto?

   (b) How large does n need to be to guarantee that, for every function $f : [n] \to [20]$, there exists some $b \in [20]$ such that $|f^{-1}(b)| \geq 3$?

3. (18 points) There are 100 pieces of candy and 35 children. Find the number of ways to distribute the candy to the children in each of the following situations.

   (a) The pieces of candy are indistinguishable and each child gets at least one piece. (The children are considered to be different from each other.)

   (b) The pieces of candy are all different and each child gets exactly one piece of candy. (So some candy is left over.)

   (c) The pieces of candy are all different but you distribute them among 35 identical paper bags assuming no bag is left empty.

4. (15 points)

   (a) By enumeration, determine $P(4, 2)$, the number of 2-partitions of 4 and $P(5, 3)$ the number of 3-partitions of 5.

   (b) For $n \geq 4$, determine a formula for $P(n, n-2)$, the number of $(n-2)$-partitions of n, and explain why your answer is correct.

5. (15 points) Give a **combinatorial** proof of the identity below.

For any positive integers m and n, $\displaystyle\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$

6. (16 points) The following question concerns the proposition below:

   Proposition: If $n \geq 1$, then $S(n, 2) = 2^{n-1} - 1$.

   (a) List all 2-partitions of [4] and show that the proposition holds for $n = 4$.

   (b) Give a **bijective** proof of the proposition. Hint: Create a bijection between the 2-partitions of $[n]$ and the nonempty subsets of $[n - 1]$.

Extra Credit: (5 points) Consider any 5 points in the xy-plane with integer coordinates. (That is, point $A(-3, 18)$ had integer coordinates but point $B(2, 4/3)$ does not.) Prove that there must exist two of the five points such that the midpoint of the line segment joining those two points also has integer coordinates.