Your Name

Solutions

Problem	Total Points	Score
1	15	
2	12	
3	16	
4	18	
5	12	
6	12	
6	15	
Extra Credit	(6)	
Total	100	

- This test is closed book.
- A student may bring one 8×11 sheet of paper with writing on the front.
- No calculator is needed since all answers can be left in "choose" form. (So $\boxed{6^8\binom{10}{4}}$ is an acceptable answer.)
- In order to receive full credit, you must **show your work.**
- Raise your hand if you have a question.

1. (15 points) After a day of skiing, a family of 4 throws their mittens into a bin. The next day, each member grabs two mittens (one right-hand and one left-hand). Count the number of ways this can happen such that not a single member of the family has both of their own mittens. (Use Inclusion-Exclusion.)

answer: We are trying to avoid having a person get both of their own mittens. So we label the family members: 1,2,3,4 and let p_i be the property that person i grabs their own pair of mittens. Thus, the number of ways the four can grab mittens such that none grab their own pair is $N_=(\emptyset)$.

We will count each "part" of the inclusion-exclusion formula separately, then put it all together at the end.

The number of ways to distribute the mittens with no restrictions is

$$N_\geq(\emptyset) = (4!)^2,$$

since there are 4! to distribute the left-hand mittens and 4! ways to distribute the right-hand mittens.

The number of ways to distribute the mittens such that person i gets their own pair is:

$$N_\geq(p_i) = (3!)^2$$

and there are 4 different choices for i.

The number of ways to distribute the mittens such that both person i and person j gets their own pair is:

$$N_\geq(p_i p_j) = (2!)^2$$

and there are $\binom{4}{2} = 6$ different choices for pairs of family members i and j.

The number of ways to distribute the mittens such that person i, person j and person k gets their own pair is:

$$N_\geq(p_i p_j p_k) = (1!)^2$$

and there are $\binom{4}{4} = 4$ different choices for i, j and k.

The number of ways to distribute the mittens such that every member of the family gets their own pair is

$$N_\geq(p_i p_j p_k p_\ell) = 1$$

and there is only one way to chose the whole family.

Now a straight application of inclusion-exclusion gives

$$N_=(\emptyset) = (4!)^2 - 4 \cdot (3!)^2 + 6 \cdot (2!)^2 - 4 \cdot (1!)^2 + 1.$$

2. (12 points)

   (a) Find the coefficient of x^{20} in $(x^3 + x^4 + x^5 + x^6 + \cdots)^4$

   Answer:

$$\begin{aligned} [\![(x^3 + x^4 + x^5 + x^6 + \cdots)^4]\!]_{x^{20}} &= [\![x^{12}(1 + x + x^2 + x^3 + \cdots)^4]\!]_{x^{20}} \\ &= [\![(1 + x + x^2 + x^3 + \cdots)^4]\!]_{x^8} \\ &= \left(\!\binom{4}{8}\!\right) = \binom{8+4-1}{8} = \binom{11}{8} \end{aligned}$$

   (b) Find $\left[\!\left[\dfrac{a}{b+cx}\right]\!\right]_{x^k/k!}$

$$\begin{aligned} \left[\!\left[\frac{a}{b+cx}\right]\!\right]_{x^k/k!} &= \frac{a}{b}\left[\!\left[\frac{1}{1-\left(\frac{-cx}{b}\right)}\right]\!\right]_{x^k/k!} \\ &= \frac{a}{b}\left(\frac{-c}{b}\right)^k k! \end{aligned}$$

3. (16 points) In each case, find a concise ordinary generating function for answering the question and also identify what coefficient you need.

   (a) How many solutions to $z_1 + z_2 + z_3 = 20$ are possible such that each z_i is an integer satisfying $1 \le z_1 \le 5$, $0 \le z_2$ and $0 \le z_3$?

   Answer: Generating function:

$$(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4 + \cdots)^2 = \frac{x(1-x^5)}{(1-x)^3}$$

   coefficient: x^{20}

   (b) How many ways to make change for a dollar using only nickels, dimes and quarters?

   Answer: Generating function:

$$(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)(1+x^{25}+x^{50}+x^{75}+\cdots) = \frac{1}{(1-x^5)(1-x^{10})(1-x^{25})}$$

   coefficient: x^{100}

4. (15 points) Solve the recurrence relation below using the generating function technique.

$$a_0 = 1 \qquad \text{and} \qquad a_n = 3a_{n-1} + 2^n, \text{ for } n \geq 1$$

Answer: Let $f(x) = \sum_{i=0}^{\infty} a_n x^n$ be the ordinary generating function for the sequence.

Using the recurrence we obtain: $a_n x^n = 3a_{n-1}x^n + 2^n x^n$, for $n \geq 1$.

Summing across all valid choices of n, we obtain

$$\sum_{i=1}^{\infty} a_i x^i = \sum_{i=1}^{\infty} 3a_{i-1}x^n + \sum_{i=1}^{\infty} 2^n x^n.$$

Using the definition of $f(x)$ to substitute in, we obtain

$$f(x) - 1 = 3xf(x) + \frac{1}{1-2x} - 1.$$

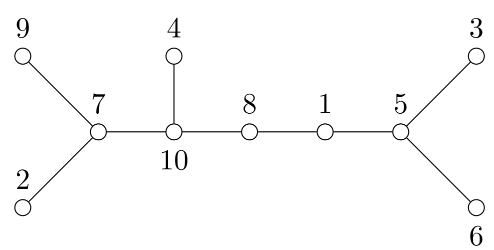
Solve for $f(x)$: $f(x) = \dfrac{1}{(1-3x)(1-2x)} = \dfrac{3}{1-3x} - \dfrac{2}{1-2x}$.

Now,

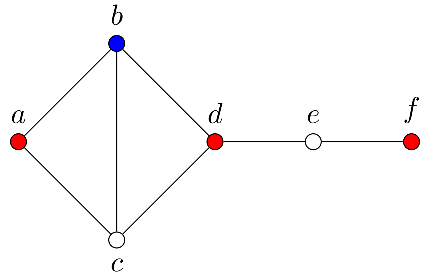
$$a_n = [\![f(x)]\!]_{a_n} = \left[\!\!\left[\frac{3}{1-3x}\right]\!\!\right]_{a_n} - \left[\!\!\left[\frac{2}{1-2x}\right]\!\!\right]_{a_n} = 3(3^n) - 2(2^n) = 3^{n+1} - 2^{n+1}.$$

5. (12 points) Let $k \geq 2$. Prove that if $\delta(G) = k$, then G contains a cycle on at least $k+1$ vertices.

6. (12 points) Draw the tree with Prüfer sequence $(7, 5, 10, 5, 1, 8, 10, 7)$.



7. (15 points) Let G be the graph pictured below.



(a) Determine the chromatic number of G. Explain your answer.

From the coloring above, we know $\chi(G) \leq 3$. On the other hand, vertices a, b and c form a K_3 and so $\chi(G) \geq 3$. Thus, $\chi(G) = 3$.

(b) Find $p(G, k)$, the chromatic polynomial of G. Explain your answer.

Claim: $p(G, k) = k(k-1)(k-2)(k-1)(k-1)(k-1)$
POC: Apply the multiplication principle to the vertices in alphabetical order. (That is, there are k choices to color vertex a and $k-1$ choices for vertex b since b must have a different color than a... and so forth.)

Extra Credit: (6 points)

1. Give an example of a graph G such that $\chi(G) = 4$ but G has no subgraph isomorphic to K_4.

Any wheel constructed from an odd cycle will do.

2. Use your answer from part a to construct a graph G such that $\chi(G) = 5$ but G has no subgraph isomorphic to K_4. (Hint: You might want to start with 4 disjoint copies of your graph from part (a).)

Let W be the wheel on 6 vertices made by joining a vertex to a 5 cycle. Make 4 copies of $W : W_1$, W_2, W_3 and W_4. Observe that for each W_i there are 6 distinct vertices and thus 6^4 ways of picking one vertex from each W_i.

Add an additional 6^4 vertices such that each is adjacent to one vertex in each W_i and each is adjacent to a different set.