1. (20 points) A soccer team will play 15 matches. The list $TWWWLWWWLWWLLTT$ is the record of a team that tied the first game, won the next three, then lost and so forth. This team ended with 8 wins, 4 losses, and 3 ties.

   (a) How many ways are there for a team to finish with 8 wins, 4 losses and 3 ties?
   $\binom{15}{8,4,3}$

   (b) How many ways in part (a) do not have consecutive losses?
   Arrange the wins and ties: $\binom{11}{8,3}$.
   Insert losses between the wins and ties. There are 12 available slots: $\binom{12}{4}$
   Ans: $\binom{11}{8,3} \cdot \binom{12}{4}$

   (c) How many ways in part (a) have a longest winning streak of 6 games.
   There must be a block of 6 W's. The remaining two W's are either together or separate. Then we proceed as in part b. That is, arrange the losses and ties, then place the blocks of wins.
   Ans: $\binom{7}{4}(8 \cdot 7 + 8 \cdot \binom{7}{2})$

2. (10 points) Draw the tree with Prüfer code $1, 4, 7, 2, 4$. Everyone got this right.

3. (10 points) Determine $\chi(G)$ and show that your answer is correct. We will show that $\chi(G) = 4$.
   Show $\chi(G) \geq 4$. Since the outer cycle has an odd number of vertices, it requires 3 colors. Since the interior vertex is adjacent to all the vertices on the outer cycle, it requires a fourth color.
   Show $\chi(G) \geq 4$. Color outer cycle 1,2,1,2,1,2,3. Color last vertex 4.

4. (10 points) Let $G = K_{r,s}$, the complete bipartite graph such that r and s are both at least 2 and assume the vertices of G are labeled.

   (a) Count the number of C_4's in G.
   This is a bipartite graph so we choose 2 vertices from the set with r vertices and 2 from the set with s vertices. There is only one way to order them in a C_4.
   Ans: $\binom{r}{2}\binom{s}{2}$

   (b) Count the number of distinct C_5's in G.
   Ans: 0. It's bipartite. No odd cycles at all.

   (c) Count the number of distinct $C_6's$ in G. Similar to part (a). Choose the 6 vertices: $\binom{r}{3}\binom{s}{3}$.
   Then count the number of ways to arrange them. Start arbitrarily at r_1. Choose the two vertices next to r_1 in $\binom{3}{2}$ ways. (Now all but two positions on the cycle are fixed.) Choose the location of r_2 in 2 ways.
   Answer: $\binom{r}{3}\binom{s}{3} \cdot 3 \cdot 2$

5. (15 points) Solve the recurrence relation below using generating functions.

$$a_0 = 1,\ a_1 = 2,\ a_n = 5a_{n-1} - 4a_{n-2} \text{ for } n \geq 2.$$

SOLUTION: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be the ordinary generating function for $\{a_k\}_{k\geq 0}$. Applying the recurrence we get: $a_n x^n = 5a_{n-1}x^n - 4a_{n-2}x^n$, for $n \geq 2$. Summing across all $n \geq 2$, we find:

$$\sum_{k\geq 2} a_n x^n = \sum_{k\geq 2} 5a_{n-1}x^n - \sum_{k\geq 2} 4a_{n-2}x^n.$$

Now, LHS= $\sum_{k\geq 2} a_n x^n = f(x) - a_0 - a_1 x = f(x) - 1 - 2x$.

Further,

$$\begin{aligned} RHS &= \sum_{k\geq 2} 5a_{n-1}x^n - \sum_{k\geq 2} 4a_{n-2}x^n \\ &= 5x \sum_{k\geq 2} a_{n-1}x^{n-1} - 4x^2 \sum_{k\geq 2} a_{n-2}x^{n-2} \\ &= 5x(f(x) - 1) - 4x^2(f(x)) \\ &= (5x - 4x^2)f(x) - 5x. \end{aligned} \tag{1}$$

Putting the LHS=RHS and solving for $f(x)$, we get: $f(x) = \frac{1-3x}{1-5x+4x^2} = \frac{1-3x}{(1-4x)(1-x)}$.

[PARTIAL FRACTIONS APPROACH] We find $\frac{1-3x}{(1-4x)(1-x)} = \frac{1/3}{1-4x} + \frac{2/3}{1-x}$. So,

$$[\![f(x)]\!]_{x^n} = \frac{1}{3}[\![\frac{1}{1-4x}]\!]_{x^n} + \frac{2}{3}[\![\frac{1}{1-x}]\!]_{x^n} = \frac{4^n + 2}{3}.$$

[CONVOLUTION APPROACH]

$$\begin{aligned} [\![f(x)]\!]_{x^n} &= [\![\frac{1}{(1-4x)(1-x)}]\!]_{x^n} - 3[\![\frac{1}{(1-4x)(1-x)}]\!]_{x^{n-1}} \\ &= \sum_{i=0}^{n} 4^i\, 1^{n-i} - 3\sum_{i=0}^{n-1} 4^i\, 1^{n-1-i} \\ &= \frac{4^{n+1}-1}{4-1} - 3\left(\frac{4^n - 1}{4-1}\right) \\ &= (4/3)4^n - (1/3) - 4^n + 1 \\ &= \frac{4^n + 2}{3} \end{aligned} \tag{2}$$

6. (20 points) In retrospect, I wish I had added the word NONTRIVIAL to both problems. I will answer with this added requirement.

   (a) Show that there exist r-regular, λ-balanced designs that are not k-uniform.

   Example: $\{1,2,3;\ 4,5,6;\ 1,4;\ 1,5;\ 1,6;\ 2,4;\ 2,5;\ 2,6;\ 3,4;\ 3,5;\ 3,6\}$ (Obviously, you can trivially construct examples with 1-blocks.)

(b) Prove that every k-uniform, λ-balanced design is r-regular. (This does require $\lambda \geq 1$ or disallowing the use of 1-blocks.)

Given a k-uniform, λ-balanced design on v varieties with b blocks, let v be a randomly chosen variety and assume it appears in n blocks. Then, the total number of pairings of v can be counted in two ways:
(1) (# of blocks containing v)(# other varieties in the block with v)=$(k-1)n$
or
(2) (# of other varieties)(# times a variety is paired with v)=$(v-1)\lambda$).
So $n = (v-1)\lambda/(k-1)$.
Since v, k, and λ are fixed, so is n.

7. (15 points) On page 158 in Theorem 4.2.8, our text proves the identity $2F_n = F_{n+1} + F_{n-2}$ for $n \geq 2$ where F_n is the nth Fibonacci number. The proof technique is induction. Prove the same identity using a combinatorial proof involving tilings of a $1 \times n$ board (and/or $1 \times (n+1)$ board and/or $1 \times (n-2)$ board) with 1- and 2-tiles.

SOLUTION

Let S be the set of all tilings of an n-board and let T be the set of all tilings of an $n+1$ board together with all tilings of an $n-2$ board.

We know $|T| = F_{n+1} + F_{n-2}$.

Now we will count T again after partitioning it into two sets: T_1 is the set of tilings of an $n+1$ board ending with a 1-tile and $T_2 = \overline{T_1}$. (Hence, T_2 contains all tilings of the $n-2$ board along with all tilings of the $n+1$ board ending in a 2-tile. Note that is is now sufficient to show that $|T_1| = |S| = F_n$ and $|T_2| = |S| = F_n$.)

Since every tiling in T_1 is a tiling of an n-board upon removal of the 1-tile, and every tiling of an n-board can be extended to a tiling of an $n+1$-board by adding a 1-tile, $|T_1| = |S| = F_n$.

Similarly, the number of tilings of an $(n-2)$-board is equal to the number of tilings of an n-board *ending in a 2-tile.* (Think of partitioning S into sets S_1 and S_2 where tilings in S_1 end with a 1-tile and tilings in S_2 end with a 2-tile. We just argued that the set of tilings of an $(n-2)$-board is equal to $|S_2.|$)

Finally, given a tiling of an $(n+1)$-board ending with a 2-tile, remove the last tile and replace it with a 1-tile to give rise to a tiling of an n-board *ending in a 1-tile.* Since this process can be reversed, the number of tilings of an $(n+1)$-board ending with a 2-tile equals the number of tilings of an n-board ending with a 1 tile (or using earlier notation $|S_1|$.)

So $F_n = |S| = |S_1| + |S_2| = |T_2.|$