Rules:
You have 60 minutes to complete the exam.
Partial credit will be awarded, but you must show your work.
No calculators, books, notes, or other aids are permitted.
If you need extra space, you can use the back sides of the pages. (Clearly label any work you want graded.)
Turn off anything that might go beep during the exam.
Good luck!

Problem	Possible	Score
1	10	
2	12	
3	12	
4	10	
5	16	
6	10	
Extra Credit	3	
Total	70	

Name: ____________________

(1) (10 points) Prove **Corollary 1.3.4:**
In a graph G, the average degree of a vertex is $\frac{2e(G)}{n(G)}$ and hence $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$.
(Make sure to address both parts of this statement.)

Pf: Let G be a graph and let $V = V(G)$, $e = e(G)$ and $n = n(G)$, $\delta = \delta(G)$, and $\Delta = \Delta(G)$.
From the Degree-Sum Theorem, we know

$$\sum_{v \in V} \deg(v) = 2 \cdot e.$$

So the average degree of G is $\frac{\sum_{v \in V} \deg(v)}{n} = \frac{2e}{n}$. *

Moreover, since $\delta \leq \deg(v) \leq \Delta$ for all $v \in V$, we know

$$\sum_{v \in V} \delta \leq \sum_{v \in V} \deg(v) \leq \sum_{v \in V} \Delta.$$

By dividing by n, we obtain:

$$\delta = \frac{n \cdot \delta}{n} = \frac{1}{n} \sum_{v \in V} \delta \leq \frac{1}{n} \sum_{v \in V} \deg v \leq \frac{1}{n} \sum_{v \in V} \Delta = \frac{n\Delta}{n} = \Delta.$$

Using *, we obtain

$$\delta \leq \frac{2e}{n} \leq \Delta.$$

(2) (12 points)

(a) Prove that if G is a graph with at least 2 vertices, deleting a vertex of degree $\Delta(G)$ **cannot** increase the average degree of G.

Pf: Let G be a graph and $v \in V(G)$ so that $\deg(v) = \Delta(G)$. Let $G' = G - v$. Then, $n \geq 2$ means the average degrees of G and G' are:

$$a = \frac{\sum_{v \in V} \deg(v)}{n} = \frac{2e(G)}{n} \quad \text{and} \quad a' = \frac{\sum_{v \in V(G)} \deg(v) - 2\Delta}{n-1} = \frac{2e(G) - 2\Delta}{n-1}.$$

$$\text{So } a' = \frac{2e(G) - 2\Delta}{n-1} = \frac{na - 2\Delta}{n-1} < \frac{(n-2)a}{n-1} < \frac{(n-1)a}{n-1} = a$$

OR an alternative approach (by contradiction) Assume $a' > a$.

Then $\frac{2e - 2\Delta}{n-1} > \frac{2e}{n}$; equivalently, $2e > 2\Delta n \geq 2 \sum_{v \in V(G)} \deg(v) = 4e$,

a contradiction if $e > 1$. If $e = 0$, then $a = a' = 0$.

(b) Prove that if G is a graph with at least 2 vertices, deleting a vertex of degree $\delta(G)$ **can** decrease the average degree of G.

Proof: Consider K_2. It has average degree 1. But $K_2 - v$ has degree 0. So its average degree decreased.

(3) (10 points) Sketch the proof of the theorem below.

Theorem 1.2.26 A graph is Eulerian if and only if has at most one nontrivial component and its vertices all have even degree.

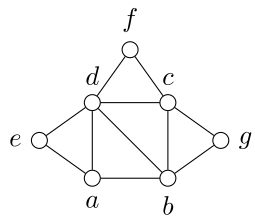
$\Rightarrow$: (easy direction) If G is Eulerian, there is a circuit containing every edge. So all edges must be in the same component.
Moreover, we can start the circuit anywhere we like, so all vertices can be viewed as interior ones. Thus, every time the circuit arrives at a vertex via edge e_i, it must leave via vertex f_i. Thus, edges incident to v can be paired up.

$\Leftarrow$: (actual work)
Use induction on the number of edges.
Base Step: $m=1$, a loop.
Inductive Step: Use earlier Lemma that states that if $\delta(G) \geq 2$, then G contains a cycle.
Find a cycle in G, say C. Then the inductive hypothesis applies to the components of G-C.
Find an Euler circuit in each component and connect all together using C.

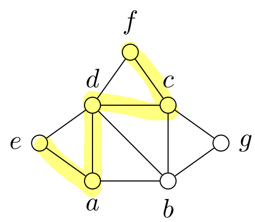
(4) (10 points) Prove **Corollary 3.1.13:** For every $k > 0$, every k-regular bipartite graph has a perfect matching.

Pf: Let G have partite sets X, Y.
Since $e(G) = k|X|$ and $e(G) = k|Y|$. We find that $|X| = |Y|$, so G is balanced.
Thus G will have a perfect matching if we can show that there is a matching that saturates X.
Thus it is sufficient to show that Hall's Theorem applies; specifically that for every $S \subseteq X$, $|S| \leq |N(S)|$.
Since G is k-regular, then the number of edges from S to N(S) is $k|S|$. On the other hand, the number of edges incident to N(S) is at most $k|N(S)|$.
Thus $k|S| \leq k|N(S)|$. Thus $|S| \leq |N(S)|$.

(5) (16 points) The graph G is below. It is copied below each questions in case you would like to draw your answer on a copy of the graph.

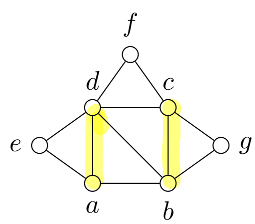


(a) Find a path in G that is maximal but not maximum or explain that one does not exist.



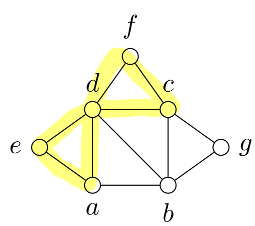
Note G has a path of Length 6.

(b) A matching in G that is maximal but not maximum or explain that one does not exist.

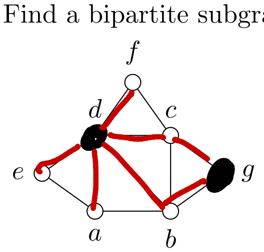


Note G has a matching of size 3.

(c) Find a circuit that is not a cycle or explain why one does not exist.



(d) Find a bipartite subgraph of G containing at least half of the edges.



$e(G) = 11$

We need at least 6 edges. They are in red.

(6) (10 points) Let v be a cut vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Pf: Let v be a cut vertex of the simple graph G. We will show that $\overline{G} - v$ is connected by finding a path between every pair of vertices of $\overline{G} - v$ based on their location in G

Case 1:

Let u, w be vertices in distinct components of $G - v$. Then $u \not\leftrightarrow w$, in G. So $u \leftrightarrow w$ in $\overline{G} - v$. So $\overline{G} - v$ contains a uw-path.

Case 2: Let u, w be vertices in the same component of $G - v$.

Since v is a cut vertex, there is a component of $G - v$ that does not contain u and w. Let z be a vertex in this component.

Thus, $u \not\leftrightarrow z$ and $w \not\leftrightarrow z$ in G. So $u \leftrightarrow z$ and $w \leftrightarrow z$ in $\overline{G} - v$. So uzw is a uw-path in $\overline{G} - v$.

Extra Credit: (4 points) Prove that every simple graph with at least two vertices has two vertices of equal degree.

Let G be a graph on n vertices.
Since G is simple, for every $v \in V(G)$,

$$\deg(v) \in \{0, 1, 2, \ldots, n-1\}.$$

But if G contains a vertex of degree 0, G cannot have a vertex of degree $n-1$. So the set of all possible degrees of G is a subset of $S_0 = \{0, 1, \ldots, n-2\}$ or $S_1 = \{1, 2, \ldots, n-1\}$, both with cardinality $n-1$.
By the PHP, two of the n vertices must have the same degree.