Rules:
You have 60 minutes to complete the exam.
Unless otherwise stated, full credit will only be given for formal proofs.
Partial credit will be awarded, but you must show your work.
No calculators, books, notes, or other aids are permitted.
If you need extra space, you can use the back sides of the pages. (Clearly label any work you want graded.)
Turn off anything that might go beep during the exam.
Good luck!

Problem	Possible	Score
1	10	
2	12	
3	12	
4	10	
5	16	
6	10	
Extra Credit	3	
Total	70	

NAME:

(1) (10 points) Prove **Corollary 1.3.4:**
In a graph G, the average degree of a vertex is $\dfrac{2e(G)}{n(G)}$ and hence $\delta(G) \leq \dfrac{2e(G)}{n(G)} \leq \Delta(G)$.
(Make sure to address both parts of this statement.)

(2) (12 points)

(a) Prove that if G is a graph with at least 2 vertices, deleting a vertex of degree $\Delta(G)$ **cannot** increase the average degree of G.

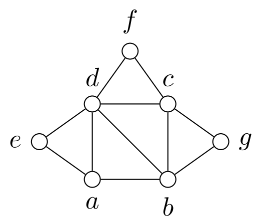
(b) Prove that if G is a graph with at least 2 vertices, deleting a vertex of degree $\delta(G)$ **can** decrease the average degree of G.

(3) (10 points) **Sketch** the proof of the theorem below. Your sketch should include the proof technique, underlying logic and any lemmas or earlier results on which the argument depends. The basic structure should be present, though some details may be omitted.

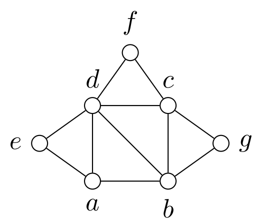
> **Theorem 1.2.26** A graph is Eulerian if and only if has at most one nontrivial component and its vertices all have even degree.

(4) (10 points) Prove **Corollary 3.1.13:** For every $k > 0$, every k-regular bipartite graph has a perfect matching.

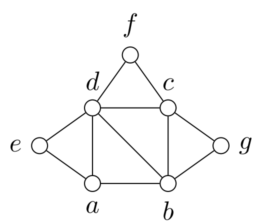
(5) (16 points) The graph G is below. It is copied below each questions in case you would like to draw your answer on a copy of the graph. Examples or short explanations are sufficient. No proofs required here.



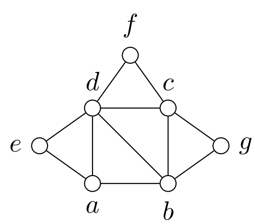
(a) Find a path in G that is maximal but not maximum or explain that one does not exist.



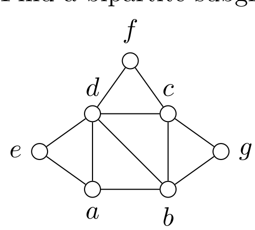
(b) A matching in G that is maximal but not maximum or explain that one does not exist.



(c) Find a circuit that is not a cycle or explain why one does not exist.



(d) Find a bipartite subgraph of G containing at least half of the edges.



(6) (10 points) Let v be a cut vertex of a simple graph G. Prove that $\overline{G} - v$ is connected.

Extra Credit: (4 points) Prove that every simple graph with at least two vertices has two vertices of equal degree.