MATH 307 FALL 2008
MIDTERM I – Selected Solutions

1. Let P be the proposition: *A sufficient condition for the stock market to fall is for winter to arrive early.*

   (a) State P as a conditional proposition. (That is, rewrite P as an If-then statement.)
   If winter arrives early, then the stock market will fall.

   (b) Write the converse of P.
   If the stock markets falls, then winter arrives early.

   (c) Write the contrapositive of P.
   If the stock market doesn't fall, then winter does not arrive early.

   (d) Write the negation of P. (Do not use the words "It is not the case that...")
   Winter arrives early and the market doesn't fall.

   (e) Which, if any, of the statements in parts b, c, and d, logically equivalent to P? The contrapositive (part c) is equivalent to P.

2. (If you have questions, ask me.)

3. (If you have questions, ask me.)

4. Use Theorem 1.1.1 Logical Equivalences to verify the logical equivalence:
   $[\sim (q \vee \sim p)] \vee (q \wedge p) \equiv p$. Supply a reason for each step.

   $$[\sim (q \vee \sim p)] \vee (q \wedge p) \equiv (\sim q \wedge p) \vee (q \wedge p) \text{ by DeMorgan's Law}$$

   $$\equiv (p \wedge \sim q) \vee (q \wedge p) \text{ by commutativity}$$

   $$\equiv p \wedge (\sim q \vee q) \text{ by the distributive law}$$

   $$\equiv p \wedge \mathbf{t} \text{ by the negation law}$$

   $$\equiv p \text{ by the identity law.}$$

5. Negate each of the following propositions.

   (a) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{Q}$ such that $\frac{x}{100} < y < x$.
   $\exists x \in \mathbb{R}\ \forall y \in \mathbb{Q}$, $\frac{x}{100} \geq y$ or $y \geq x$.

   (b) $\forall x \in \mathbb{Z}$, if $x \geq 10$ and x is prime, then $x + 2$ is not prime or $x + 4$ is not prime.
   $\exists x \in \mathbb{Z}$, such that $x \geq 10$ and x is prime and $x + 2$ is prime and $x + 4$ is prime.

(c) $\forall x \in \mathbb{R}$, $|x| < 1$ if and only if $x^2 < 1$.
$\exists x \in \mathbb{R}$, ($|x| < 1$ and $x^2 \geq 1$) or ($x \geq 1$ and $x^2 < 1$.)

6. Determine the truth value for each of the following and justify your answer.

   (a) For every composite number c, $c^2 \geq 16$.
   True. By definition, the first composite number is 4. That is, for every composite number c, $c \geq 4$. Now for every real number, if $c \geq 4$, then $c^2 \geq 16$.

   (b) $\forall x \in \mathbb{R}$ if x^2 is even, then x is even.
   False. Let $x = \sqrt{2}$. Then $x \in \mathbb{R}$ and $x^2 = 2$ which is even and x is not even since it is not an integer. So, $x = \sqrt{2}$ is a counterexample.

   (c) $\forall x \in \mathbb{R}$ such that $x \neq 0$, $\exists y \in \mathbb{R}$ such that $xy > 0$.
   True. Given any real number x not equal to zero, choose $y = x$. Then $xy = x^2 \geq 0$ because any real number squared is nonnegative. Furthermore, since $x \neq 0$, $x^2 \neq 0$, by the zero property. Thus, for any given x, we have a choice of y such that xy is always positive.

7. (a) Define what it means for the integer a to be divisible by the integer b.
   Look in your book.

   (b) Use the definitions (of divisibility and odd) to prove that, for any two consecutive odd integers, the difference of their squares is a multiple of 8. (Note: for any two numbers n and m the *difference of their squares* means $n^2 - m^2$.)

   Proof: Let n and m be consecutive odd integers. So, $n = m + 2$. Also, from the definition of odd, we know there exists an integer k such that $m = 2k + 1$. Thus, by substitution, $n = 2k + 3$. Now, $n^2 - m^2 = (2k + 3)^2 - (2k + 1)^2 = 8k + 8 = 8(k + 1)$. Let $k_1 = k + 1$. Since k is an integer, k_1 is an integer. Thus, $n^2 - m^2 = 8k_1$ where k_1 is an integer. Thus, by the definition of divides, we have shown that 8 divides $n^2 - m^2$. Or, equivalently, we have shown that $n^2 - m^2$ is a multiple of 8 for any pair of consecutive odd integers m and n.