Math 307

Exam II - Solutions

(1) (20 points) Let $A = \{1, 2, 3\}$, $B = \{x \in \mathbb{R} | 0 \leq x \leq 2\}$, and $C = \{x \in \mathbb{R} | -1 \leq x \leq 3\}$. Assume the universe is the set of real numbers.The answers to the first four questions are not unique.

(a) Give an example of a proper, nonempty subset of A.
$\{1\}$

(b) List two distinct elements of the set B.
1,2

(c) List two distinct elements of the set $\mathcal{P}(B)$, the power set of B.
$\{1\}, \{2\}$

(d) Give an example of a nonempty set disjoint from B.
$\{3\}$

(e) Find $B \triangle C$.
$[-1, 0) \cup (2, 3]$

(f) Find $B \cap A$.
$\{1, 2\}$

(g) Find $B \cup A$.
$[0, 2] \cup \{3\}$

(h) Find $\overline{B}$.
$(-\infty, \infty)$

(i) Find $A - C$.
$\emptyset$

(j) Find $\overline{A} \cup \overline{B}$.
Using the fact that $\overline{A} \cup \overline{B} = \overline{A \cap B}$, we get $\{x \in \mathbb{R} | x \neq 1 \wedge x \neq 2\}$

(2) (20 points) Let $A = \{1, 2, 3, 4, \cdots, 29, 30\}$.

(a) How many distinct subsets of A are possible?
2^{30}

(b) How many distinct subsets of A have exactly eight elements?
$\binom{30}{8}$

(c) Assuming all subsets of A are equally likely, what is the probability of picking a subset with exactly eight elements?
$\binom{30}{8}/2^{30}$

(d) How many eight element subsets contain both 1 and 2 or contain both 3 and 4?
$\binom{28}{6} + \binom{28}{6} - \binom{26}{4}$

(3) (10 points) Use the laws of set theory to simplify

$$\overline{(\overline{A - B}) \cap A}.$$

$\overline{(\overline{A - B}) \cap A} = \overline{(\overline{A - B})} \cup \overline{A} = (A - B) \cup \overline{A} = (A \cap \overline{B}) \cup \overline{A} = (A \cup \overline{A}) \cap (\overline{B} \cup \overline{A}) = \mathcal{U} \cap (\overline{B} \cup \overline{A}) = (\overline{B} \cup \overline{A})$

(4) (10 points) Write and label the converse, inverse, and contrapositive of the implication:

If it is Saturday and there is snow, Tom goes skiing.
converse: If Tom goes skiing, then it's Saturday and there's snow.
inverse: If it is not Saturday or there isn't snow, then Tom doesn't ski
contrapositive: If Tom isn't skiing, then it isn't Saturday or there isn't snow.

(5) (15 points) Assume the universe is the set of real numbers. Determine whether the following propositions are true or false. Carefully explain your answers using complete sentences.

(a) $\forall x \ \exists y \ \ 0 < x - y < 1$

Answer: True

For any x, pick $y = x - 0.5$. Then, $x - y = 0.5$ which is strictly between 0 and 1. So for all x we found a y that makes the inequality true.

(b) $\exists x \ \ \forall y \quad [(x < y) \rightarrow (y^2 > 3)]$

Answer: True

I'll pick $x = \sqrt{3}$. (You could pick x to be any number greater than or equal to $\sqrt{3}$.) Then for any $y \leq \sqrt{3}$, the hypothesis is false, thus the implication is true. For any $y > \sqrt{3}$, $y^2 > 3$ and the implication is true. Thus we have found an x such that the implication is true for all y.

(6) (10 points) Negate and simplify the following:

$$\forall x \quad \left[(x > 0) \rightarrow (\exists y \ \ 0 < y < \sqrt{x})\right]$$

$\neg\left[\forall x \quad [(x > 0) \rightarrow (\exists y \ \ 0 < y < \sqrt{x})]\right]$
$\Leftrightarrow \exists x \quad \neg\left[[(x > 0) \rightarrow (\exists y \ \ 0 < y < \sqrt{x})]\right]$
$\Leftrightarrow \exists x \quad [(x > 0) \vee \neg(\exists y \ \ 0 < y < \sqrt{x})]$
$\Leftrightarrow \exists x \quad [(x > 0) \vee (\forall y \ \ 0 \geq y \vee y \geq \sqrt{x})]$

(7) (15 points) Establish the validity of the argument below by listing a series of numbered steps and reason for the steps.

$a \rightarrow (b \rightarrow c)$
$d \vee a$
$\neg d \rightarrow b$
$\underline{\neg d \qquad\quad}$
$\therefore c$

Steps	Reasons
(1) $d \vee a$	premise
(2) $\neg d$	premise
(3) a	Rule of Disjunctive Syllogism applied to (1) and (2)
(4) $\neg d \rightarrow b$	premise
(5) b	Rule of Detachment applied to (2) and (4)
(6) $a \wedge b$	Rule of Conjunction applied to (5) and (3)
(7) $a \rightarrow (b \rightarrow c)$	premise
(8) c	Rule of Conditional Proof applied to (7) and (6)