MATH 307 | Quiz #10 |
DISCRETE MATH | §6.8, 7.1 | SPRING 2016

NAME: Solutions

This quiz contains 4 problems worth 30 points. You may not use books, notes, or a calculator. You do not have to simplify your answers. You have 30 minutes to take the quiz.

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1. (8 points) Suppose that eleven distinct integers are selected from the set $\{1, 2, 3, \cdots, 19, 20\}$. Prove that at least two of the eleven have a sum equal to 21.

Place elements into pairs that sum to 21:

1,20; 2,19; 3,18; 4,17; 5,16; 6,15; 7,14; 8,13; 9,12; 10,11;

There are 10 such pairs.
Let X = be the 11 selected and Y = the set of 10 pairs that sum to 21. Let $f: X \to Y$ be the function that maps the selected number to the pair that contains it. Since $\lceil |X|/|Y| \rceil = \lceil \frac{11}{10} \rceil = 2$, the PHP implies that two distinct selections are mapped to the same pair. Thus, this pair sums to 21.

2. (8 points) An inventory consists of a list of 200 items, each marked "available" or "unavailable." There are 103 available items. Show that there are at least two available items in the list exactly 5 items apart.

Let $X_1 = \{x_1, x_2, \ldots, x_{103}\}$ be the positions in the list of the available items. Let $X_2 = \{x_1+5, x_2+5, \ldots, x_{103}+5\}$ be the positions 5 ahead of the available items. Let $Y = \{1, 2, \ldots, 205\}$.
Let $f: X_1 \cup X_2 \to Y$ by $f(x) = x$. Since $\left\lceil \frac{|X_1 \cup X_2|}{|Y|} \right\rceil = \left\lceil \frac{206}{205} \right\rceil = 2$, the PHP implies two elements of $X_1 \cup X_2$ must be mapped to the same number in Y.
Since the x_i are all distinct, the pair mapped to the same number in Y must have the form $x_i = x_j + 5$. Thus the available items at x_i and x_j are listed 5 apart.

3. Assume a person deposits \$200 into an account at the beginning of *each year* and that the account earns 10% interest compounded annually. Assume no money is withdrawn from the account. Let A_i denote the amount in the account at the end of i years.

   (a) (3 points) Find A_i for $i = 1, 2, 3$. (Actually do the calculation. It isn't hard.)

$A_1 = (200)(1.10) = 220$

$A_2 = (220+200)(1.10)$
$= 420 + 42 = 462$

$A_3 = (462+200)(1.10)$
$= 662 + 66.20$
$= 728.20$

answer
$A_1 = 220$
$A_2 = 462$
$A_3 = 728.20$

   (b) (4 points) Find a recurrence relation for A_n.

$A_n = (200 + A_{n-1})(1.10)$ for $n \geq 1$ and $A_0 = 0$

4. (7 points) Suppose that we have n dollars and that each day we buy either coffee (\$1), tea (\$1), a cookie (\$2), a bagel (\$3), or a burrito (\$3). Let R_n be the number of ways of spending all the money. Derive a recurrence relation for the sequence $R_1, R_2, R_3, \cdots$ [Assume order is taken into account. So the \$4 purchase (coffee, coffee, cookie) is different from the purchase (coffee, cookie, coffee). Also make sure you include appropriate and complete initial conditions.]

R_1 = # ways to spent \$1 = 2 (coffee or tea)

R_2 = # ways to spend \$2 = 5

list: cookie, cc, ct, tc, tt

R_3 = # ways to spend \$3 = 14

list: bagel, burrito, (use k=cookie, c=coffee, t=tea)
kc, kt, tk, tc,
$2^3 = 8$ for ccc, cct, ctc, tcc, and so forth

$8+4+2 = 14$

answer:

$R_1 = 2$

$R_2 = 5$

$R_3 = 14$

for $n \geq 4$,

$R_n = 2R_{n-1} + R_{n-2} + 2R_{n-3}$