MATH 307 Quiz #7
DISCRETE MATH §4.3 SPRING 2016

NAME: Solutions

This quiz contains 4 problems worth 30 points. You may not use books, notes, or a calculator. You have 30 minutes to take the quiz.

Theta Form	Name		Theta Form	Name
$\theta(1)$	Constant		$\theta(n^2)$	Quadratic
$\theta(lg(lg(n)))$	Log log		$\theta(n^3)$	Cubic
$\theta(lg(n))$	Log		$\theta(n^k)$, $k \geq 1$	Polynomial
$\theta(n)$	Linear		$\theta(c^n)$, $c > 1$	Exponential
$\theta(n\, lg(n))$	n log n		$\theta(n!)$	Factorial

Fact from Calc 2:
$1 + a + a^2 + \cdots a^k = \frac{a^{k+1}-1}{a-1}$

Before reading any further, it will be helpful to walk through the strategies being applied and observe that there are only THREE of them. Also keep in mind that in real life, the justification is really important. You can't look your answer up in the back of the book. Here are the strategies:

1. *In every case,* to show $f(n) = \theta(g(n))$, I will give explicitly constants C_1 and C_2, so that
$$C_1 g(n) \leq f(n) \leq C_2 g(n).$$

2. For any sum of positive numbers, I find the lower bound by deleting some of the terms. For example, I obtain a lower bound for $1+2+4+\cdots 2^{n-1} + 2^n$ by deleting $1+2+4+\cdots+2^{n-1}$ as seen below:
$$2^n \leq \underbrace{1+2+4+\cdots 2^{n-1}}_{\text{delete these to get}} + 2^n.$$

3. For any sum, I find the upper bound by replacing terms by ones at least as large. For example, I obtain an upper bound for $8\lg n + 3n + 5$ by replacing $8 \lg n$ by $8n$ and replacing 5 by $5n$ as seen below:
$$8\lg n + 3n + 5 \leq 8n + 3n + 5n$$
replace replace!

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1. (3 points) Fill in the blank below in the definition:

For $f(n)$ and $g(n)$ be functions with domain $\{1, 2, 3, \cdots\}$, we write $f(n) = O(g(n))$ if $\exists\, C \in \mathbb{R}^+$ so that $|f(n)| \leq C\,|g(n)|$ for all but finitely many $n \in \mathbb{Z}^+$.

2. (12 points) Select a theta notation from the table for each expression and justify your answer.

(a) $5\lg n + 3n^2 + 2n\lg n$

Ans : $\Theta(n^2)$

Justification :

$$\underbrace{3}_{c_1} n^2 \le \underbrace{5\lg n + 3n^2 + 2n\lg n}_{f(n)} \le 5n^2 + 3n^2 + 2n^2 = \underbrace{10}_{c_2} n^2$$

lower bound f(n) upper bound

(b) $3 + 6 + 9 + 12 + \cdots + (3n)$ Ans : $\Theta(n^2)$

Justification : $3+6+9+\cdots+3n = 3(1+2+3+\ldots+n) = 3 \cdot \frac{n(n+1)}{2} = \frac{3}{2}(n^2+n)$.

Now,

$$\underbrace{\frac{3}{2}}_{c_1} n^2 \le \underbrace{\frac{3}{2}(n^2+n)}_{f(n)} \le \frac{3}{2}(n^2+n^2) = \underbrace{3}_{c_2} n^2$$

lower f(n) upper

(c) $1 + 2 + 4 + 8 + \cdots + 2^n$

$$\underbrace{2^n}_{c_1 = 1} \le \underbrace{1+2+4+8+\ldots+2^n}_{f(n)} = \frac{2^{n+1}-1}{2-1} = 2^{n+1} - 1 \le 2^{n+1} = \underbrace{2}_{c_2} \cdot 2^n$$

lower f(n) upper

3. (8 points) Answer the questions using the algorithm below:

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i = n
while (i > 1){
   for j = 1 to i
      x = x + 1
   i = ⌊i/2⌋
}
```

(a) Give an expression (in terms of n) for the exact number of time the statement $x = x+1$ is evaluated.

$n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \dots + \frac{n}{2^k}$ where $k = \lfloor \lg n \rfloor$

(b) Select a theta notation from those in the table for the number of times the statement $x = x + 1$ is evaluated and justify your answer.

$$n \leq n + \frac{n}{2} + \frac{n}{4} + \dots + \frac{n}{2^k} = n\left(1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^k}\right) = n\left(\frac{\left(\frac{1}{2}\right)^{k+1} - 1}{\frac{1}{2} - 1}\right)$$

$$= 2n\left(1 - \left(\frac{1}{2}\right)^{k+1}\right) \leq 2n$$

So $f(n) = \Theta(n)$

4. (7 points) Show that if $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

If $f(n) = O(g(n))$, then there is $C \in \mathbb{R}^+$ so that $|f(n)| \leq C|g(n)|$.
Since $C \neq 0$ and $C > 0$, we can divide by C and leave the inequality fixed to get: $\frac{1}{C}|f(n)| \leq |g(n)|$ where $\frac{1}{C} \in \mathbb{R}^+$.

Thus, $g(n) = \Omega(f(n))$