Your Name (print clearly)

Solutions

Monday 28 March 2016

Page	Total Points	Score
1	19	
2	29	
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5	10	
extra credit	5	
Total	100	

Instructions and information:

- Please turn off cell phones or any other thing that will go BEEP.
- Scientific calculators are allowed on this test. You may not use a cell phone or a laptop.
- Read the directions for each problem. You must always show your work to receive partial credit.
- Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- If you need more room, use the backs of the pages and indicate to the grader where to look.
- Raise your hand (or come up to the front) if you have a question.
- Formulas from Calculus:

  1. $1 + a + a^2 + \cdots + a^k = \frac{a^{k+1}-1}{a-1}$
  2. $\log_a n = \log_b n / \log_b a$
  3. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

1. (a) (4 points) Fill in the blanks below:

   i. A function f from X to Y is *one-to-one* if $\forall y \in Y$ there is at most one $x \in X$ so that $(x,y) \in f$.

[alt. def: If $f(x_1)=f(x_2)$, then $x_1=x_2$]

   ii. A function f from X to Y is *onto* if $\forall y \in Y$ there is at least one $x \in X$ so that $(x,y) \in f$.

[alt. def: the range of f is Y.]

   (b) (4 points) Assume that the domain and the codomain of the function $f(n) = \lfloor n/4 \rfloor$ is the set of all integers. Determine if $f(n)$ is one-to-one, onto, or both.

not one-to one : $f(0)=\lfloor 0 \rfloor = 0$, $f(1) = \lfloor 1/4 \rfloor = 0$.

f is onto : For every $z \in \mathbb{Z}$, let $x = 4z$.

Then $f(x) = \left\lfloor \frac{4z}{4} \right\rfloor = z$.

   (c) (3 points) Given an example of a function $g(x)$ from $\mathbb{R}$ to $\mathbb{R}$ that is one-to-one but not onto and show that your example is not onto. (You do not need to justify the one-to-one property.)

$f(x) = e^x$; Since $e^x > 0$ for all real x, $f(x) = 0$ has no solution.

2. (8 points) Let a be a sequence defined by $a_n = 3n + 1$, for $n = 1, 2, 3, \cdots$.

   (a) Find a_6. $3 \cdot 6 + 1 = 17$

   (b) Find $\sum_{i=1}^{3} a_i$

$a_1 + a_2 + a_3 = 4 + 7 + 10 = 21$

   (c) Find a formula for the subsequence of a obtained by selecting every other term of a starting with the first.

We want index $n = 2m-1$ for $m = 1, 2, 3, \ldots$

So $a_1, a_3, a_5, \ldots$ becomes $b_1, b_2, b_3, \ldots$ where $b_m = 3(2m-1)+1$

That is: $\boxed{b_m = 6m - 2 \text{ for } m = 1, 2, 3, \ldots}$

3. (14 points) Let R be a relation on the set of positive integers defined by $(x, y) \in R$ if 2 divides $x + y$.

 (a) Find an example of an ordered pair in R. $(1,1)$ [$2=1+1$ is divisible by 2]

 (b) Find an example of an ordered pair not in R. $(1,2)$ [$3=1+2$ is not divisible by 2]

 (c) Is R reflexive? Yes

 $x+x=2x$ which is divisible by 2. So $\forall x \in \mathbb{Z}, (x,x) \in R$.

 (d) Is R antisymmetric? No.

 $(1,3)$ and $(3,1) \in R$ since $1+3=3+1=4$.

 (e) Is R transitive? Yes.

 If $(x,y),(y,z) \in R$, then $(x+y)+(y+z)=2s+2r$. So $x+z=2s+2r-2y$.

 So $2 \mid x+z$. So $(x,z) \in R$.

4. (15 points) Let S be the set of all strings on the set $\{0, 1, 2\}$ of length 3 or less. Let R be the equivalence relation on S defined by $s_1 R s_2$ if the strings s_1 and s_2 have the same number of zeros.

 (a) Explain why the string 12 is related to the string 211.

 Both strings have zero 0's.

 (b) Explain why the string 120 is not related to the string 001.

 120 has one 0, but 001 has two 0's.

 (c) Find all elements in [001], the equivalence class containing the string 001.

 $[001] = \{$ all strings w/ two 0's $\} = \{001, 010, 100, 002, 020, 200\}$

 (d) How many equivalence classes does R have?

 4, because a string can have 0, 1, 2 or 3 zeros.

 (e) List one member of each equivalence class.

 1, 01, 001, 000

5. (a) (7 points) Fill in the blank below in the definition:

For $f(n)$ and $g(n)$ be functions with domain $\{1, 2, 3, \cdots\}$, we write $f(n) = O(g(n))$

if $\exists C \in \mathbb{R}$ so that $|f(n)| \leq C|g(n)|$ for all but finitely many $n \in \mathbb{Z}^+$.

(b) Use the definition above to show $1 + 2^k + 3^k + \cdots + n^k$ is $O(n^{k+1})$.

$$1+2^k+3^k+\dots+n^k \leq n^k+n^k+\dots+n^k \quad \text{since } i^k \leq n^k \text{ for all } 0 < i \leq n.$$
$$= n \cdot n^k$$
$$= 1 \cdot n^{k+1}.$$

6. (12 points) For each expression below, select a θ notation from the table and justify your answer.

Theta Form	Name	Theta Form	Name
$\theta(1)$	Constant	$\theta(n^2)$	Quadratic
$\theta(lg(lg(n)))$	Log log	$\theta(n^3)$	Cubic
$\theta(lg(n))$	Log	$\theta(n^k)$, $k \geq 1$	Polynomial
$\theta(n)$	Linear	$\theta(c^n)$, $c > 1$	Exponential
$\theta(n\ lg(n))$	n log n	$\theta(n!)$	Factorial

(a) $\frac{n^3+5n\lg n}{4n+8}$

denominator larger

make denominator smaller

$$\frac{1}{12}n^2 = \frac{n^3}{12n} = \frac{n^3}{4n+8n} \leq \frac{n^3}{4n+8} \leq \frac{n^3+5n\lg n}{4n+8} \leq \frac{n^3+5n^3}{4n+8} \leq \frac{6n^3}{4n} = \frac{3}{2}n^2$$

constant

numerator smaller

replace terms in numerator w/ larger terms

Constant

$$\text{So } \frac{n^3+5n\lg n}{4n+8} = \theta(n^2)$$

(b) $3 + 9 + 27 + \cdots + 3^n$.

$$1 \cdot 3^n \leq 3+9+27+\dots+3^n = 3(1+3+9+\dots+3^{n-1})$$

delete first n-1 terms

$$= 3\left(\frac{3^n-1}{3-1}\right) = \frac{3}{2}(3^n-1)$$

use formula

$$\leq \frac{3}{2}\cdot 3^n$$

add $\frac{3}{2}$ (or drop "-1")

$$\text{So, } 3+9+\dots+3^n = \theta(3^n)$$

7. (8 points) Let $m = 2^3 \cdot 5 \cdot 11^2 \cdot 17^3$ and $n = 2^2 \cdot 7 \cdot 11^5$.

(a) Find the greatest common divisor of m and n.

$\gcd(m,n) = 2^2 \cdot 11^2$

(b) Find the least common multiple of m and n.

$\mathrm{lcm}(m,n) = 2^3 \cdot 5 \cdot 7 \cdot 11^5 \cdot 17^3$

8. (15 points) Let $m = 159$ and $n = 509$.

(a) Trace the Euclidean Algorithm for inputs m and n above. (You need to show your steps, at least in abbreviated form and state the output explicitly.)

E. Alg

$509 = 3 \cdot 159 + 32$
$159 = 4 \cdot 32 + 21$
$32 = 1 \cdot 21 + 11$
$21 = 1 \cdot 11 + 10$
$11 = 1 \cdot 10 + 1$
$10 = 10 \cdot 1 + 0$

return 1

work for part (c) here

$509 - 3(159) = 32$
$159 - 4(32) = 21$
$32 - 21 = 11$
$21 - 11 = 10$
$11 - 10 = 1$

$$\begin{array}{r} 14 \\ 3 \\ \hline 42 \end{array}$$

(b) What is the significance of the number returned by the Euclidean Algorithm?

It returns the greatest common divisor of its inputs.

(c) Write the greatest common divisor of m and n as a linear combination of m and n.

So $1 = 11 - 10$
$= 11 - (21 - 11)$
$= 2 \cdot 11 - 21$
$= 2(32 - 21) - 21$
$= 2 \cdot 32 - 3 \cdot 21$
$= 2 \cdot 32 - 3(159 - 4 \cdot 32)$
$= 14 \cdot 32 - 3 \cdot 159$
$= 14(509 - 3 \cdot 159) - 3(159)$
$= 14 \cdot 509 - 45(159)$

ANS: $1 = 14(509) - 45(159)$

9. (10 points)

(a) Write the decimal number 900 in binary.

$900 = 2^9 + 2^8 + 2^7 + 2^2$

So in binary the decimal number 900 would be

$\underbrace{1\,1}\,\underbrace{1\,0\,0\,0}\,\underbrace{0\,1\,0\,0}$

(b) Write the decimal number 900 in hexadecimal.

using the collections of 4 from above:

384

Extra Credit (5 points): Prove that $\lg(n!) = \theta(n \lg n)$.

$O(n\lg n)$: $\lg(n!) = \lg 1 + \lg 2 + \ldots + \lg n \leq \lg n + \lg n + \ldots + \lg n = n(\lg n)$ ✓

$\Omega(n\lg n)$: $\lg(n!) = \lg 1 + \lg 2 + \ldots + \lg\lceil\frac{n}{2}\rceil + \lg(\lceil\frac{n}{2}\rceil+1) + \ldots + \lg n$

$\geq \lg\lceil\frac{n}{2}\rceil + \lg(\lceil\frac{n}{2}\rceil+1) + \ldots + \lg n$

$\geq \lg\lceil\frac{n}{2}\rceil + \lg(\lceil\frac{n}{2}\rceil) + \ldots + \lg\lceil\frac{n}{2}\rceil$

$= \lceil\frac{n+1}{2}\rceil \cdot \lg\lceil\frac{n}{2}\rceil$

$\geq \frac{n}{2} \cdot \lg\frac{n}{2} = \frac{n}{2}(\lg n - \lg 2) = \frac{n}{4}\lg n + (\frac{n}{4}\lg n - \lg 2)$

$\geq \frac{n}{4}\lg n$ ✓