1. (10 points) Use $A = \{1, 3\}$, $B = \{2, 3, 4\}$, and universal set is $U = \{0, 1, 2, 3, 4, 5\}$ to find the quantities below.

   (a) $\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{3\}, \{1, 3\}\Big\}$

   (b) $B - A = \{2, 4\}$

   (c) $\overline{(\overline{A} \cup B)}$

   First: $\overline{A} = \{0, 2, 4, 5\}$. So, $\overline{A} \cup B = \{0, 2, 3, 4, 5\}$. Finally $\overline{(\overline{A} \cup B)} = \{1\}$

2. (10 points) Use $X = \{1, 2, \emptyset, \{1, 2, 3\}, \mathbb{Z}\}$ to answer the following questions.

   (a) Find $|X|$.

   $|X| = 5$

   (b) Is $1 \in X$? Explain.

   Yes. 1 is the first element listed in X.

   (c) Is $3 \in X$? Explain.

   No. The elements of X are 1, 2, the empty set, the set $\{1, 2, 3\}$, and the set of integers. The number 3 did not appear in this list.

   (d) Is $\{1, 2, 3\} \subseteq X$? Explain.

   No. While $1 \in X$ and $2 \in X$, as seen in part (c), $3 \notin X$. Thus, $\{1, 2, 3\} \not\subseteq X$. As an aside, it would be correct to say that $\{1, 2, 3\} \in X$ or $\Big\{\{1, 2, 3\}\Big\} \subseteq X$.

3. (10 points) For the propositions below, propositions p and q are true, but the truth value of proposition r is unknown. Determine whether each proposition below is True, False, or has unknown status at this time. Justify your answer.

   (a) $p \leftrightarrow (q \rightarrow r)$

   This has unknown truth value. If $r = F$, the proposition is false. If $r = T$, the proposition is true.

   (b) $\neg(p \vee r) \wedge q$

This is false. Since $p = T$, we know $(p \vee r) = T$. Thus, $\neg(p \vee r) = F$. Thus the conjunction is false.

4. (10 points) Determine if the argument below is valid. Justify your answer. (Note that should you choose to use the truth table, you must explain what about your table shows that the argument of valid or not.)

$$\begin{array}{c} p \vee q \\ (p \wedge q) \to r \\ q \wedge \neg r \\ \hline \therefore \neg p \end{array}$$

p	q	r	$p \vee q$	$(p \wedge q) \to r$	$q \wedge \neg r$	$\neg p$
T	T	T	T	T	F	
T	T	F	T	F		
T	F	T	T	T	F	
T	F	F	T	T	F	
F	T	T	T	T	F	
F	T	F	T	T	T	T
F	F	T	F			
F	F	F	F			

This argument is VALID. We see from the truth table that row 6 is the only row in which all the hypotheses are true and, in this case, the conclusion is true.

If you want to make a general argument, among other things, you need words, sentences, and clear connectors. Here is such an argument.

We claim the argument is valid. We must show that all true hypotheses force a true conclusion. Since hypothesis 3, $\boxed{q \wedge \neg r}$, would have to be true, we conclude $q = T$ and $r = F$. Thus, hypothesis 1 $\boxed{p \vee q}$ is true. Since hypothesis 2, $\boxed{(p \wedge q) \to r}$, would have to be true and we know $r = F$, we know $p \wedge q = F$. But we know $q = T$. Thus, $p \wedge q = F$ only if $p = F$. Thus, the conclusion $\neg p = T$.

5. (10 points) Write the converse and the contrapositive of the following:

The team wins if the quarterback can run.

First I rewrite the sentence above is standard form:

If the quarterback can run, then the team wins.

Converse: If the team wins, then the quarterback can run.

Contrapositive: If the team does not win, then the quarterback does not run.

6. (10 points) Write the negation of the proposition $\forall x \Big((P(x) \vee Q(x)) \to \neg R(x) \Big)$

Answer: $\exists x\Big((P(x) \vee Q(x))\Big) \wedge R(x)$

7. (10 points) Write down two distinct propositions which are logically equivalent to $\neg p \to q$.

   (a) $p \wedge \neg q$

   (b) $\neg(\neg(p \to q)) \equiv \neg(p \wedge \neg q) \equiv \neg p \vee q$

8. (10 points) Determine the truth value of each proposition below and justify your answer.

   (a) $\forall x \in \mathbb{Z},\ \forall y \in \mathbb{Z}\ \big((x < y) \to (x^2 < y^2)\big)$

   False. It is sufficient to find a counter example. Let $x = -2$ and $y = -1$. Then $-2 = x < y = -1$, but $4 = x^2 > 1 = y^2$.

   (b) $\forall v \in \mathbb{R}^+,\ \exists u \in \mathbb{R}^+$ such that $\frac{u+1}{v} > 2$

   True. Given any v, let $u = 2v + 1$. Note that since $v > 0$, $2v + 1 = u > 0$.

   Now $\frac{u+1}{v} = \frac{2v+2}{v} = 2 + \frac{2}{v} > 2$.

9. (10 points) Prove the following statement. For any integer n, if n^3 is an even integer then n is even. (You must state the method of proof you are using: direct, contradiction, contrapositive.)

Proof by contrapositive is the easiest.

Proof: We will prove the contrapositive: For any integer n, if n is odd, then n^3 is odd.

Assume n is an odd integer. Then, using the definition of odd, $n - 2k + 1$ for some integer k. Now $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^2 + 6k + 3k) + 1 = 2k_1 + 1$, where $k_1 = 4k^2 + 6k + 3k \in \mathbb{Z}$. Thus, we have shown that n^3 is odd.

10. (10 points) Prove using mathematical induction that for all $n \geq 1$,

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Proof: Let $S(n)$ be the statement that $1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2}$.

Basis Step: We must show that $S(1)$ is true. But $S(1)$ is the statement: $3{\cdot}1 - 2 = 1 = (1{\cdot}2)/2$, which is true.

Inductive Step: We assume $S(n)$ is true. That is, we assume $1+4+7+\cdots+(3n-2) = \frac{n(3n-1)}{2}$.

We must show $S(n+1)$ is true. That is, we must show that $1+4+7+\cdots+(3n-2)+(3n+1) = \frac{(n+1)(3n+2)}{2}$.

Now,

$$\begin{aligned}
\text{LHS} &= \big(1+4+7+\cdots+(3n-2)\big)+(3n+1) && \text{the definition of LHS of } S(n+1)\\
&= \tfrac{n(3n-1)}{2}+(3n+1) && \text{by the inductive hypothesis}\\
&= \tfrac{3n^2-n+6n+2}{2} && \text{common denominator}\\
&= \tfrac{(3n+2)(n+1)}{2} && \text{factoring}\\
&= \text{RHS}
\end{aligned}$$

Thus, if $S(n)$ is true, then $S(n+1)$ is true. Thus we have shown that $S(n)$ is true for all $n \geq 1$.

Extra Credit (5 points) Prove that for any real number $x > -1$ and any positive integer n, $(1+x)^n \geq 1+nx$. (Full credit will be given only for proofs that use the assumption $x > -1$.)

Proof: We will proceed by induction on n. So $S(n)$ is the statement that $(1+x)^n \geq 1+nx$.

Basis Step: Let $n = 1$. Then $S(1)$ is the statement that $(1+x)^1 = 1+x = 1+1\cdot x$, which is certainly true.

Inductive Step: Assume the statement $S(n)$ is true for some n. That is, we assume

$$(**) \qquad (1+x)^n \geq 1+nx.$$

We must show that $S(n+1)$ is true. That is, we must show $(1+x)^{n+1} \geq 1+(n+1)x$ is true.

Since $x > -1$, we know that $1+x > 0$. Thus, we can multiply both sides of the inequality (**) above by $1+x$ and not change the direction of the inequality to get:

$$(1+x)^n(1+x) \geq (1+nx)(1+x).$$

Now we have $(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$, because $nx^2 > 0$.

Thus, if $S(n)$ is true, then $S(n+1)$ is true. Thus we have shown that $S(n)$ is true for all $n \geq 1$.