Mon 11 Sept

- Hmk 2 due Fri.
- Hmk guidelines posted on Hwk page.
- My solns + .tex files for HWK 1 posted in Canvas @ HW1 assignment link.
- Class notes posted in Canvas, ~~not on public webpage~~.

  webpage

Proof Writing Guidelines

1. You need to prove your assertions. If you are asked to count the. number of edges in a graph, you must give the count **and** justify the count. If you are asked to construct an example, you must describe its construction and demonstrate that it satisfies the necessary requirements.

2. Use precise technical language.

   For example, instead of referring to vertices as having "connections" say that the vertices are "adjacent". Be wary of making up your own secret lingo.

3. Be wary of "it", "that", "this" as mechanisms to refer to previous parts of your argument.

4. Leave lots of space between problems for my comments.

5. Always include the problem statement.

6. Always begin a sentence with a word in English, never with a symbol.

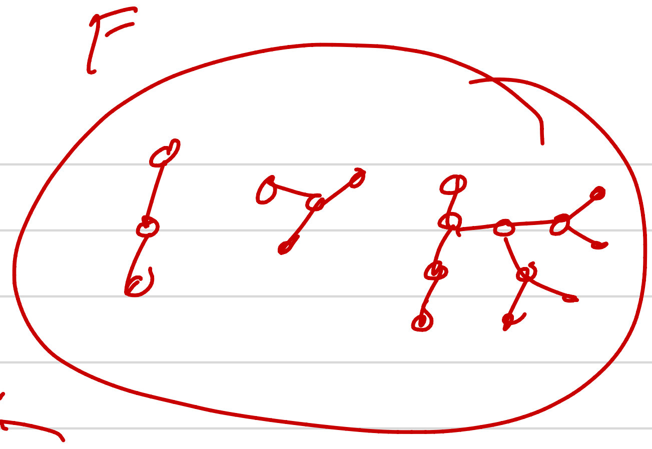
   For example, the sentence "$d(x) > k$ by definition." is bad. Change it to: "By definition, $d(x) > k$."

7. Do not replace words with symbolic logic.

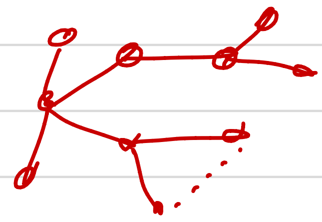
   For example, do not write "$x > y$ and $y > 0 \Longrightarrow x^2 > y^2$." Instead write, "Since $x > y > 0$, it follows that $x^2 > y^2$."

   Another example, do not write "We know that $|S| >$ the number of vertices in the largest component of G." Instead, write "...$|S|$ is greater than ..."

§1.5 Trees and Forests

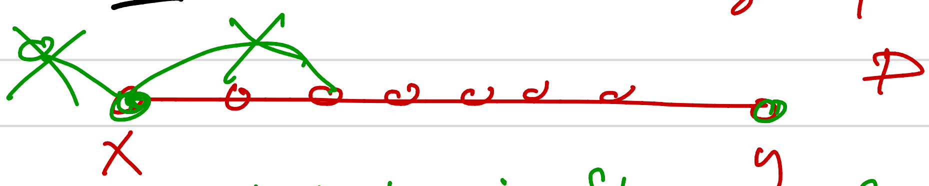


- Forest: an acyclic graph
- Tree: a connected forest
- Leaf: a vertex of degree 1.



Class Prop: Every tree of order ≥ 2 has at least two leaves.

Pf: Let P be a longest path in tree T.



$(|T| \geq 2 \Rightarrow P$ has distinct end points$)$

* Conseq: Induction is often a useful proof strategy. B/c if T is a tree $\wedge$ v is a leaf on T then $T-v$ is tree.

Thm 1.5.1 TFAE

(i) T is a tree

(ii) Any two vertices of T are linked by a unique path

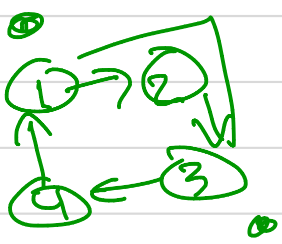
(iii) T is minimally connected. (ie: Every edge is a bridge.)

(iv) T is maximally acyclic. ($\forall e \notin T$, $T+e$ contains a cycle)

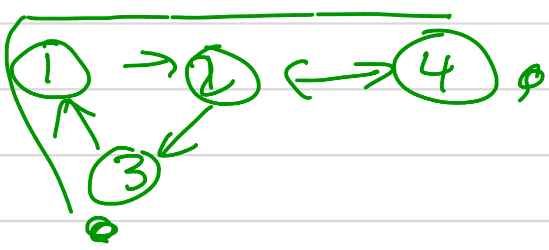
* How do you structure such a proof?

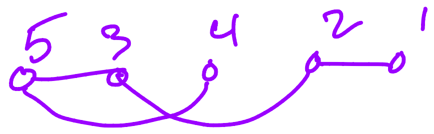
technically one must prove (A) $\Leftrightarrow$ (B) six times

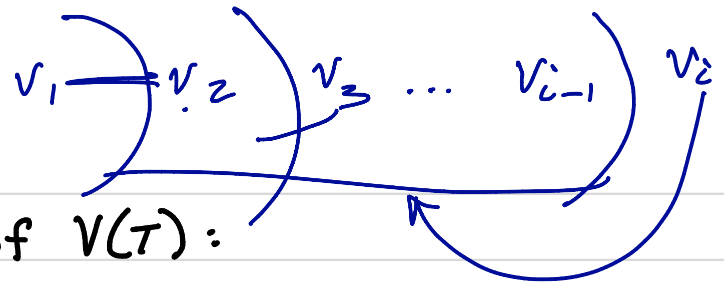
In practice: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)



(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\wedge$ (ii) $\Leftrightarrow$ (iv)







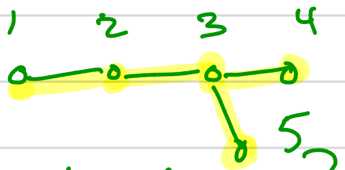
Cor 1.5.2 : T tree.

$\exists$ an ordering of $V(T)$:

$v_1, v_2, \dots, v_n$

such that $\forall i$, $2 \le i \le n$,

v_i has a unique neighbor in $\{v_1, \dots, v_{i-1}\}$.



Good ordering: (5, 3, 4, 2, 1)

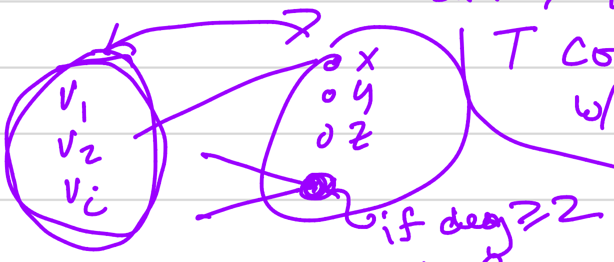
$G_n[v_1 v_2 \dots v_n] = G$

Pf : By Prop 1.4.1, we can order the vertices of T such that w/ ordering $(v_1, v_2, \dots, v_n)$

$G_i := G[v_1, v_2, \dots, v_i]$ is connected.

an induced subgraph

How do we know $\exists$ v_{i+1} w/ precisely 1 nbh in G_i?



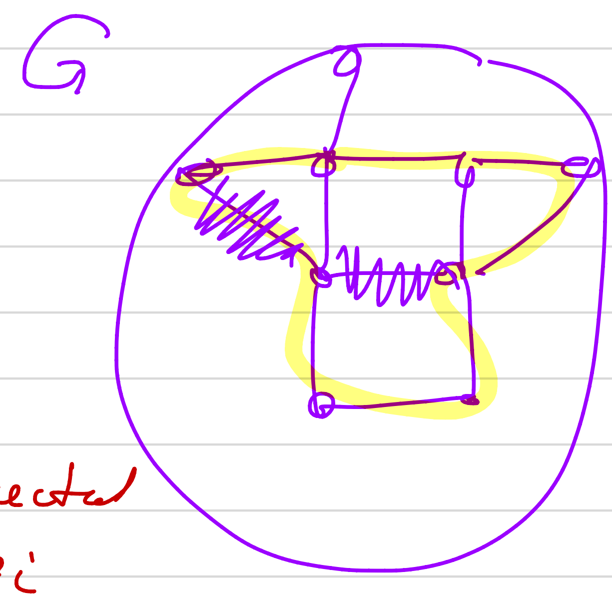
T connect $\Rightarrow \exists$ some vertex w/ some edge to G_i.

T acyclic $\wedge$ G_i connected $\Rightarrow$ $\exists$ exactly 1 edge.

if deg ≥ 2 to G_i $\Rightarrow$ cycle. Impossible!

Cor 1.5.3 : Let G be a graph on n vertices.

$\boxed{G \text{ is a tree}} \iff \boxed{G \text{ is connected and } |E(G)| = n-1}$



Proof : G has n vert.

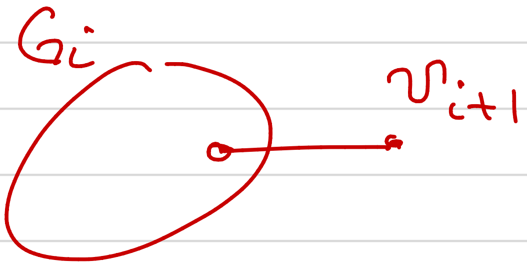
($\Rightarrow$): Spps G is a tree. So G is connected by def.

Nts. $|E(G)| = n-1$. ~~(by induction on n)~~

Apply 1.5.2 and order vert of G:

$v_1, v_2 \dots v_n$ s.t. $G_i = G[v_1, \dots, v_i]$ is connected

and s.t. v_{i+1} has exactly 1 nbh to G_i



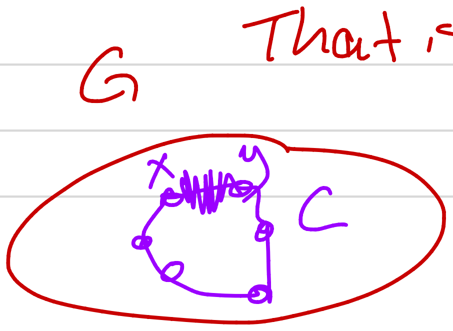
Then, G_n was obtained by adding 1 edge for each i for $i \ge 2$.

And $G_n = G$. So $E(G) = n-1$.

($\Leftarrow$): Sppse. G connected $\wedge$ $\boxed{|E(G)| = n-1}$. Show G is a tree.

Nts G is acyclic. Strategy: by contradiction.

That is: Spps G is connected $\wedge$ $|E(G)| = n-1$ $\wedge$ G has a cycle.



Delete $xy \in E(G)$. Repeat iteratively. Stop when an acyclic, connected graph is obtained aka a tree. $\Rightarrow\Leftarrow$ b/c part 1

$T \leq G$

$n = |V(T)|$

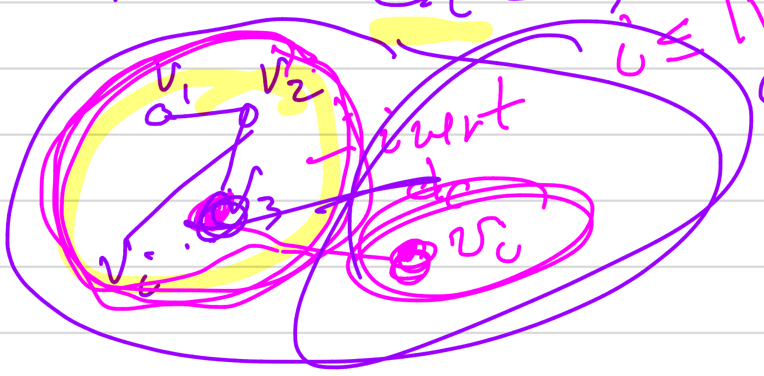
Cor 1.5.4: T tree, G graph s.t. $\delta(G) \geq |V(T)| - 1$.

Then $T \leq G$

Pf: Return to ordering verts as in Cor 1.5.2: $v_1, v_2, \dots, v_n$

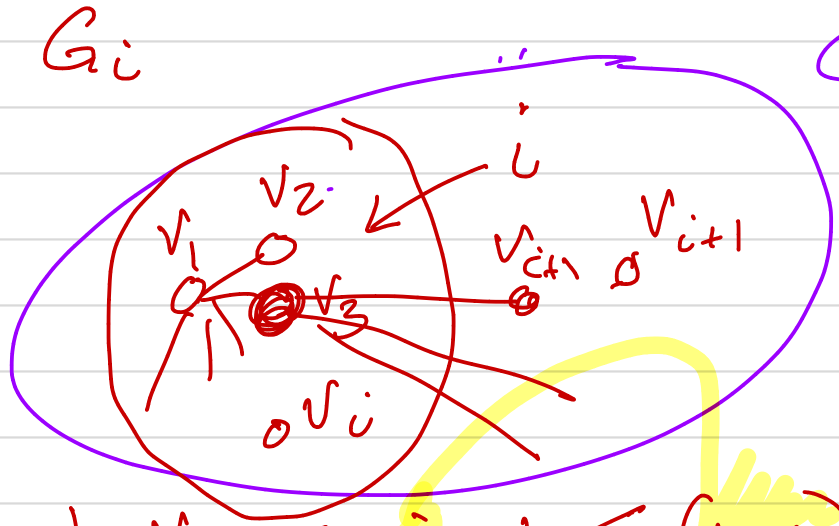
Find G_1 in G

Assume $G_i \leq G$, $i \leq |V(T)| - 1$



$T: v_1, v_2, \dots, v_n$

G_i



$d_{G_i} v_3 \leq i - 1 \leq (n-1) - 1$

$d_G(v_3) \geq \delta(G) \geq |V(T)| - 1$

$d_{G_i}(v_3) \leq i - 1 \leq |V(T)| - 1 - 1 \leq |V(T)| - 2 < |V(T)| - 1$

def: tree T is a <u>spanning tree</u> of graph G if T is a tree, $T \leq G$, and $V(T) = V(G)$

Fact: Every connected graph G contains a spanning tree.

POF: