Disclaimers: If a definition, term, or notation was discussed in class and/or appeared on the homework, you are expected to know it. There is no claim that this review is perfect.

Chapter 4: Planar Graphs

- terms: plane graph, face, outer face, outer planar, maximally planar, plane triangulation, maximal plane graph. planar graph
- theorems to remember:
  - Thm 4.4.1 Jordan Curve theorem
  - Prop 4.2.4: A plane forest has exactly one face.
  - Prop 4.2.6: In a 2-connected plane graph, every face is bounded by a cycle.
  - Prop 4.2.8 A plane graph on at least three vertices is maximally plane if and only if it is a plane triangulation.
  - Cor 4.2.10 A plane graph has at most $3n-6$ edges (provided $n \geq 3$). Every plane triangulation with n vertices has exactly $3n-6$ edges.
  - Cor 4.2.11 A plane graph contains neither a K^5 nor a $K_{3,3}$ as a subgraph.
  - Prop 4.4.1 Every maximal plane graph is maximally planar. For a planar graph, maximally planar is equivalent to having $3n-6$ edges (provided $n \geq 2$).
- theorems to know by name: Thm 4.2.9 Euler's Formula, Thm 4.4.6 Kuratowski's Theorem A graph is planar if and only if it has no K^5 or $K_{3,3}$ minor.

Chapter 5: Coloring

- terms: coloring, vertex coloring, edge coloring, k-coloring, k-edge-coloring, k colorable, k-edge colorable, k-chromatic, k-edge-chromatic, chromatic number, edge chromatic number, $\chi(G)$, $\chi'(g)$, greedy coloring, Mycielski's construction , color class
- theorems to remember:
  - Lemma 5.2.3 Every k-chromatic graph contains a subgraph of minimum degree at least $k-1$.
  - Prop 5.3.1 If G is bipartite, then $\chi'(G) = \Delta(G)$.
- theorems to know by name:
  - Them 5.2.4 Brook's Theorem Let G be a connected graph. Then $\chi(G) \leq \Delta(G)$ or G is a complete graph or G is an odd cycle.
  - Thm 5.3.2 Vizing's Theorem For every (simple) graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.

Chapter 6: Flows

- terms: network, capacity, flow, integral flow $\overrightarrow{E}(G)$, cut, $\overrightarrow{E}(X,Y)$ $\overrightarrow{e}$, $\overleftarrow{e}$, $c(X,Y)$, $f(X,Y)$, value of a flow, $|f|$, capacity of a cut.
- theorems to remember:

  - Prop 6.2.1: In a network N with cut S, $f(S,\overline{S}) = f(s,V)$.

- theorems to know by name: Thm 6.2.2 Ford Fulkerson In every network, the maximum value of a flow is equal to the minimum capacity of a cut.

Chapter 7: Extremal Graph Theory

- terms: Turán graph, extremal graph, extremal number, $ex(n,H)$
- theorems to know by name: Thm 7.1.1 Turán For all integers r and n with $r > 1$, if G is K^r-free and $|E(G)| = ex(n,K^r)$, then $G \cong T^{r-1}(n)$.

Fri 10 Nov

- Hwk 9 due today
- Mid 2 ← FBKS
  next Fri
  4pm - 6pm
- Adjusted end of semester schedule.
- Agenda
  - Prove Ramsey's Thm formally
  - Review
  - Other?

Thm 9.1.1

$\forall r \in \mathbb{N}, \exists n \in \mathbb{N}$ s.t. every graph on at least n vertices contains K^r or $\overline{K^r}$ as an induced subgraph.

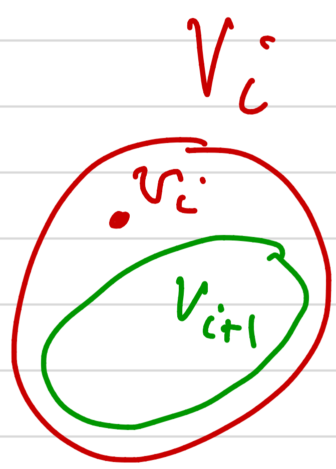
Pf: ① $r=1$ ✓ K^1 •
② $r \geq 2$

$n = 2^{2r-3}$, G graph w/ at least n vertices

Claim We can construct the following nested subsets $V_1, V_2, \dots, V_{2r-2} \subseteq V$ and find $v_1, v_2, \dots, v_{2r-2}$, $v_i \in V_i$ satisfying

① $|V_i| = 2^{2r-2-i}$ for $i \in \{1, 2, \dots, 2r-2\}$ ✓

② $V_i \subseteq V_{i-1} - \{v_{i-1}\}$ $i \in \{2, \dots, 2r-2\}$



③ v_{i-1} is either adj to all vert. in V_i or is nonadj to all vert in V_i. $i \in \{2, 3, \dots, 2r-2\}$

pick as the larger of $N_{V_{i-1}}(v_{i-1})$ or

$V_{i-1} - N_{V_{i-1}}(v_{i-1})$

PoC: Choose V_1 arb. 2^{2r-3} subset of V.

Sppse chosen $V_1, V_2, \dots, V_{i-1}$ as in the claim. Describe/check, we can construct V_i.

Know $|V_{i-1}| = 2^{2r-2-(i-1)} = 2^{2r-1-i}$

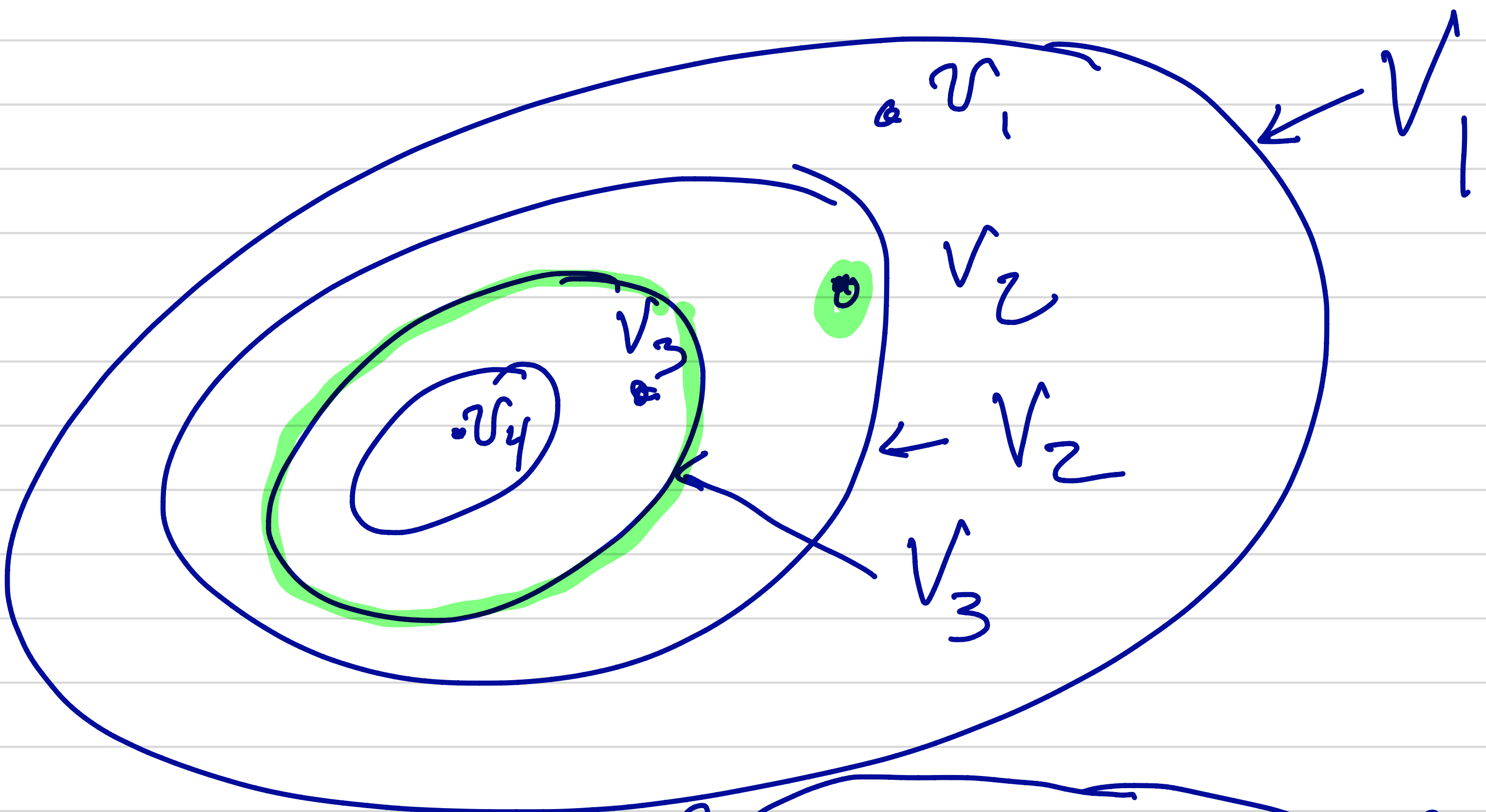
Know $|V_{i-1}| = 2^{2r-2-(i-1)} = 2^{2r-1-i}$

So $|V_{i-1} - \{v_{i-1}\}| = 2^{2r-1-i} - 1 \leftarrow$ odd

So $|V_i| \geq \frac{1}{2} |V_{i-1} - \{v_{i-1}\}| = \frac{1}{2}(2^{2r-1-i} - 1)$

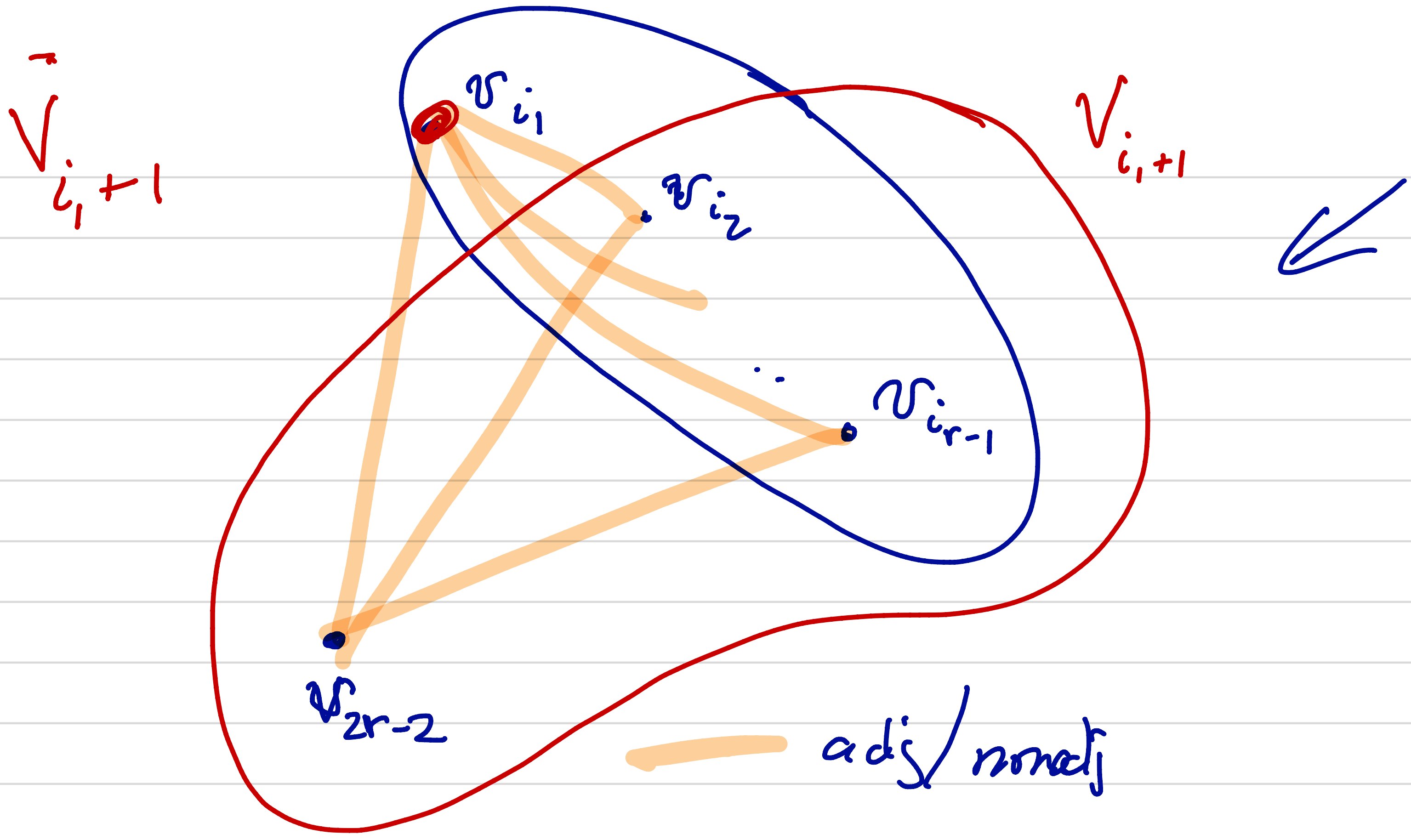
$|V_i| = 2^{2r-1-i-1}$

We are choosing V_i s.t. the vertex v_{i-1} is either adj to all or nonadj to all vert in V_i. $\boxtimes$



Consider the view of $v_1, v_2, \ldots, v_{2r-3}$ from v_{2r-2}. It has the same view (adj / nonadj) to at least half of $v_1, v_2, \ldots, v_{2r-3}$. So at least $\frac{2r-3}{2} = r - \frac{3}{2} \stackrel{\text{discrete}}{=} r-1$ have same view.

$2r-3 = 17 \quad \frac{17}{2} = 8.5$



$\underbrace{v_1, v_2, v_3 \ldots v_{2r-3}}$ v_{2r-2}

have selected r-1 vertices

v_{i_1} v_{i_2} .. $v_{i_{r-1}}$

$i_1 \in \{1, 2, .., 2r-3\}$

$i_1 < i_2$

$v_{i_j} \in V_{i_k+1}$ for $k < j$

So the rel. between v_{i_k} and v_{i_j} and v_{2r-2} must be the same.

Ramsey Theory Notation

$$R(r) = \min\{n \in \mathbb{N} : \text{every } n\text{-vertex graph contains } K^r \text{ or } \overline{K^r} \text{ as a subgraph}\}$$

<u>Ramsey's Thm</u> $R(r)$ finite

Proof of R's Thm : $R(r) \le 2^{2r-3}$

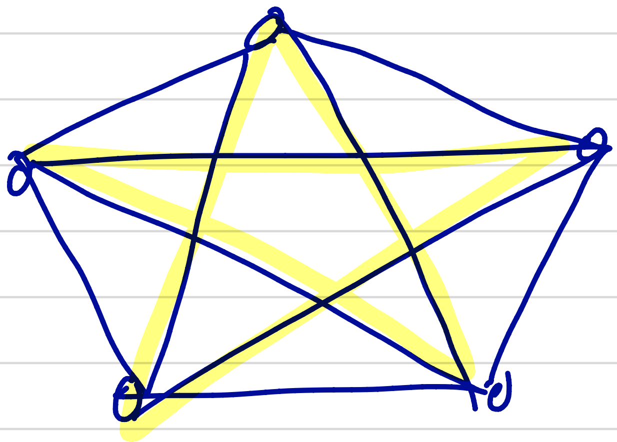
$$R(r) = \underbrace{R(K^r, K^r)} = \underline{R(K^r, K^r; 2)}$$

$$R(r) = R(K^r, K^r)$$

$$= \min\{n \in \mathbb{N} : \text{every 2-coloring of the edges of a } K^n \text{ has either a red } K^r \text{ or a blue } K^r\}$$

$$R(K^3, K^3) \leq 2^{2\cdot r-3} = 2^{2\cdot 3-3} = 2^3$$
$$= 8$$

$$\boxed{6 \leq R(K^3, K^3) = 6}$$



$$2^{r/2}$$