Fri 13 Oct

- Hmwk #6 due Fri
- Midterms returned by Mon (?)
- ~~Mon notes + video posted~~

- Agenda for today
  - recall planar stuff
  - Euler's Formula
  - soften up Kuratowski

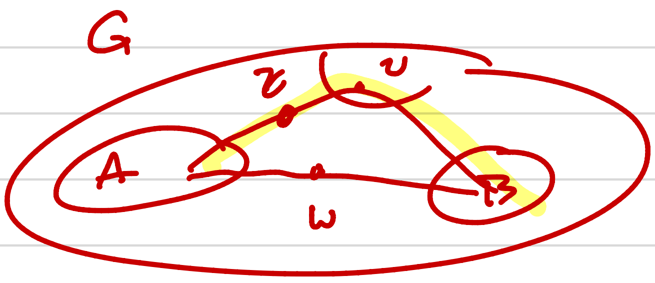
G is 2-conn.

$\forall v \in V\ \exists w \in N(v)$ s.t.

$G - v - w$ still conn

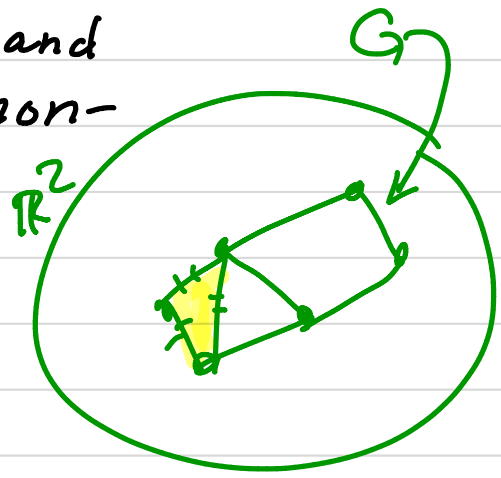
$\exists v$ s.t. ($\neg\ \forall w \in N(v)$)

- $G - v - w$ disconn
- w is a cutvertex in $G - v$



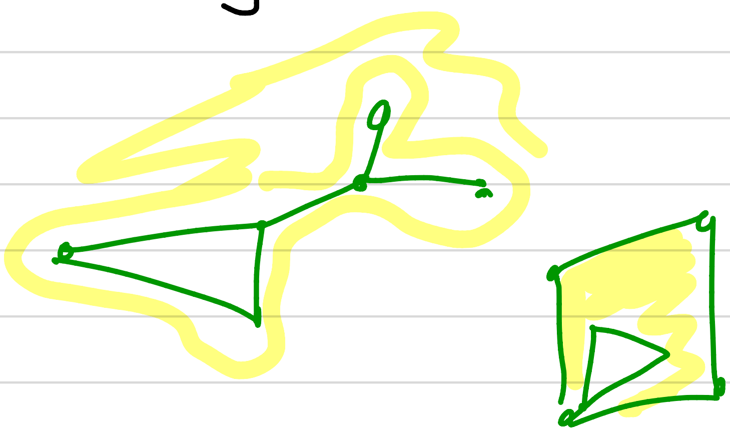
From Monday

- $G = (V, E)$ is a plane graph means $V, E \subseteq \mathbb{R}^2$ s.t.
  V - points in $\mathbb{R}^2$
  E - arcs in $\mathbb{R}^2$ made of finite # of straight line segments and different edges have non-intersecting arcs.

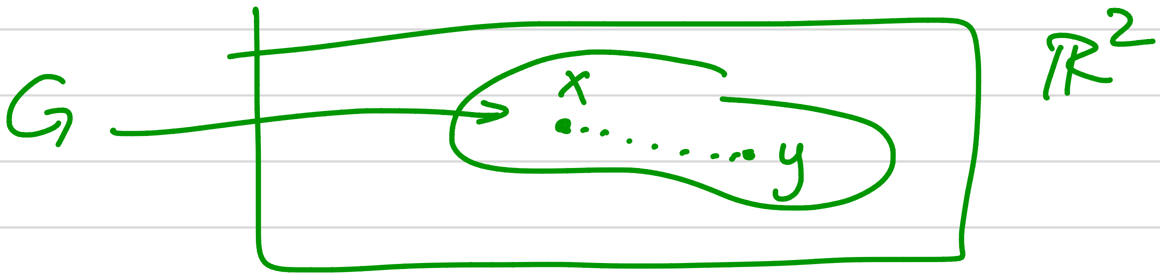


- faces of plane graph G are the (open) sets in $\mathbb{R}^2 - G$ with boundary G.

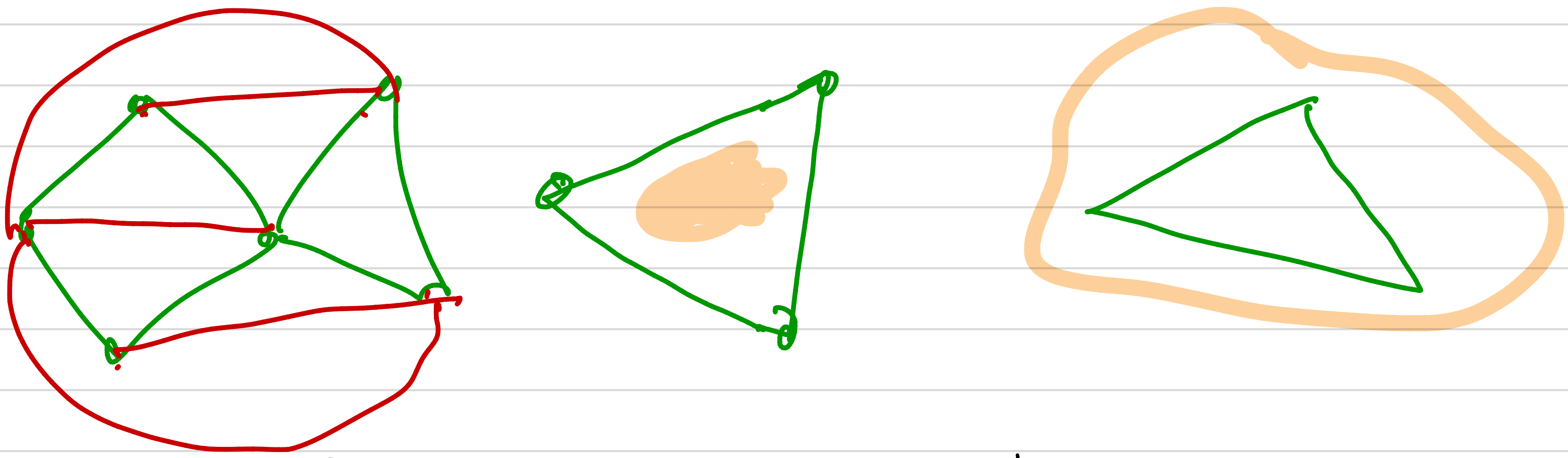
- If G is a 2-connected plane graph then the boundary of every face is a cycle.



def: • plane graph $G=(V,E)$ is called maximally plane (or maximal) if $\forall e \in \overline{G}$, it is not possible to add e to G and the result still be plane.



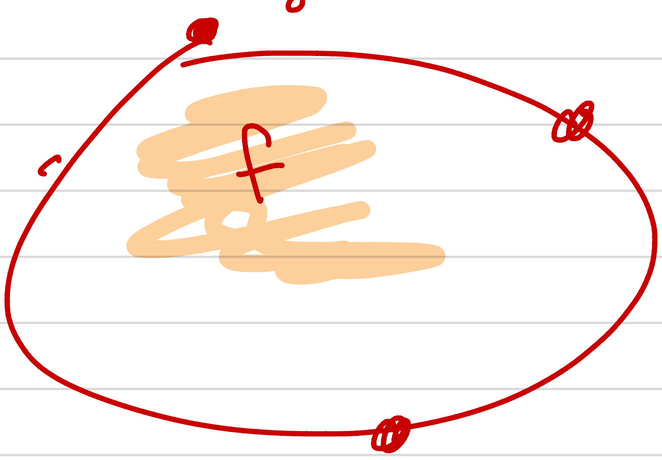
• G is a plane triangulation if every face of G is bounded by K^3.



Prop 4.2.8 G is a plane graph on at least 3 vertices

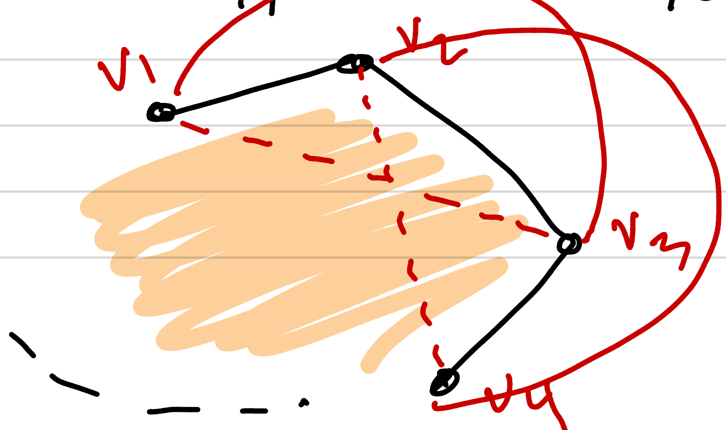
G is maximal $\iff$ G is a plane triangulation.

Pf: $\Leftarrow$: Spps G is a triangulation. NTS no added edge is possible.
Any added edge is an arc w/i 1 face



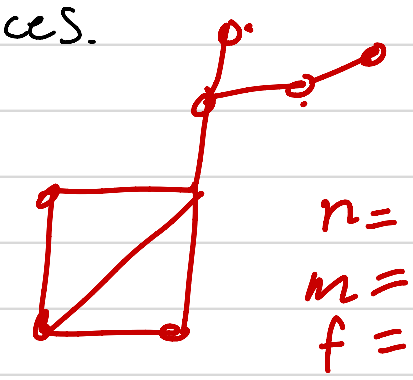
Since each face is K^3, no edges to add.

$\Rightarrow$: Spps G maximal plane graph. Then no face can have 4 vertices. So all faces are K^3.



Euler's Formula (Thm 4.2.9)

G connected plane graph s.t. $n = \#$ vertices
$m = \#$ edges
$f = \#$ faces.



$n = 8$
$m = 9$
$f = 3$
$n - m + f = 8 - 9 + 3 = 2$

then $\boxed{n - m + f = 2}$

Pf: Sppse $|V| = n$, fixed.
Induction on $m = |E|$.

Base $m = n-1$ (G is a tree)
So, $f = 1$

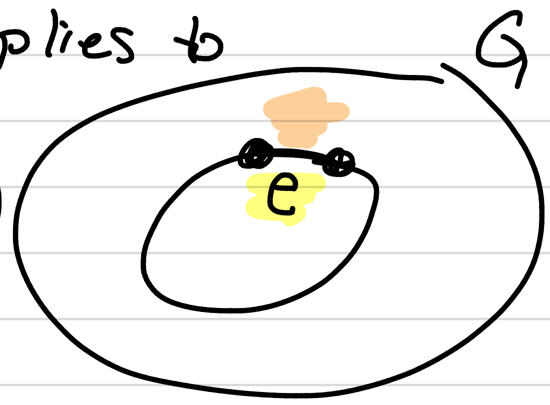
$$n - m + f = n - (n-1) + 1 = 2 \checkmark$$

In. Step: Ind. Hypoth: Every plane graph on n vertices & fewer than m edges satisfies E's Form.
Spps G is a plane graph on $m > n-1$ edges.

B/C $m \geq n$, G has a cycle. So pick e on a cycle in G. So $G - e$ is connected & has fewer edges. So I.hyp. applies to

$G^* = G - e$.

$n^* = n$
$m^* = m - 1$
$f^* = f - 1$



By the I.hyp: $n^* - m^* + f^* = 2$

$$n - (m-1) + (f-1) = 2$$
$$n - m + f = 2 \checkmark$$

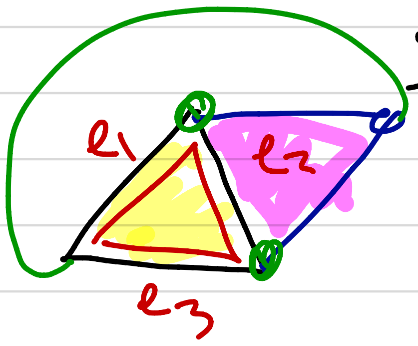
Cor 4.2.10: • G plane graph on n vertices, $n \geq 3$
then $\|G\| \leq 3n-6$.

Every plane triangulation has exactly $3n-6$ edges.
maximally plane graph

Pf: Suff to show Δulation has $3n-6$ edges.

G is plane triangulation $\Rightarrow$ # face, boundary is K^3.

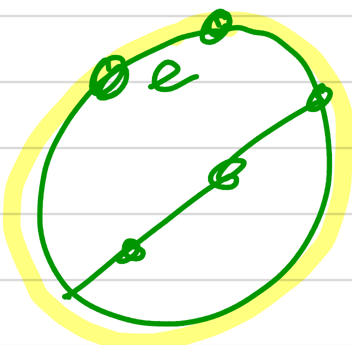
If f = # faces, then count edges by



$3 \cdot f$. But this double counts all edges.

So $3 \cdot f = 2 \cdot m$ or $f = \frac{2}{3} m$

$2 = n - m + f = n - m + \frac{2}{3} m$
$6 = 3n - 3m + 2m = 3n - m$
$m = 3n - 6$ ✓

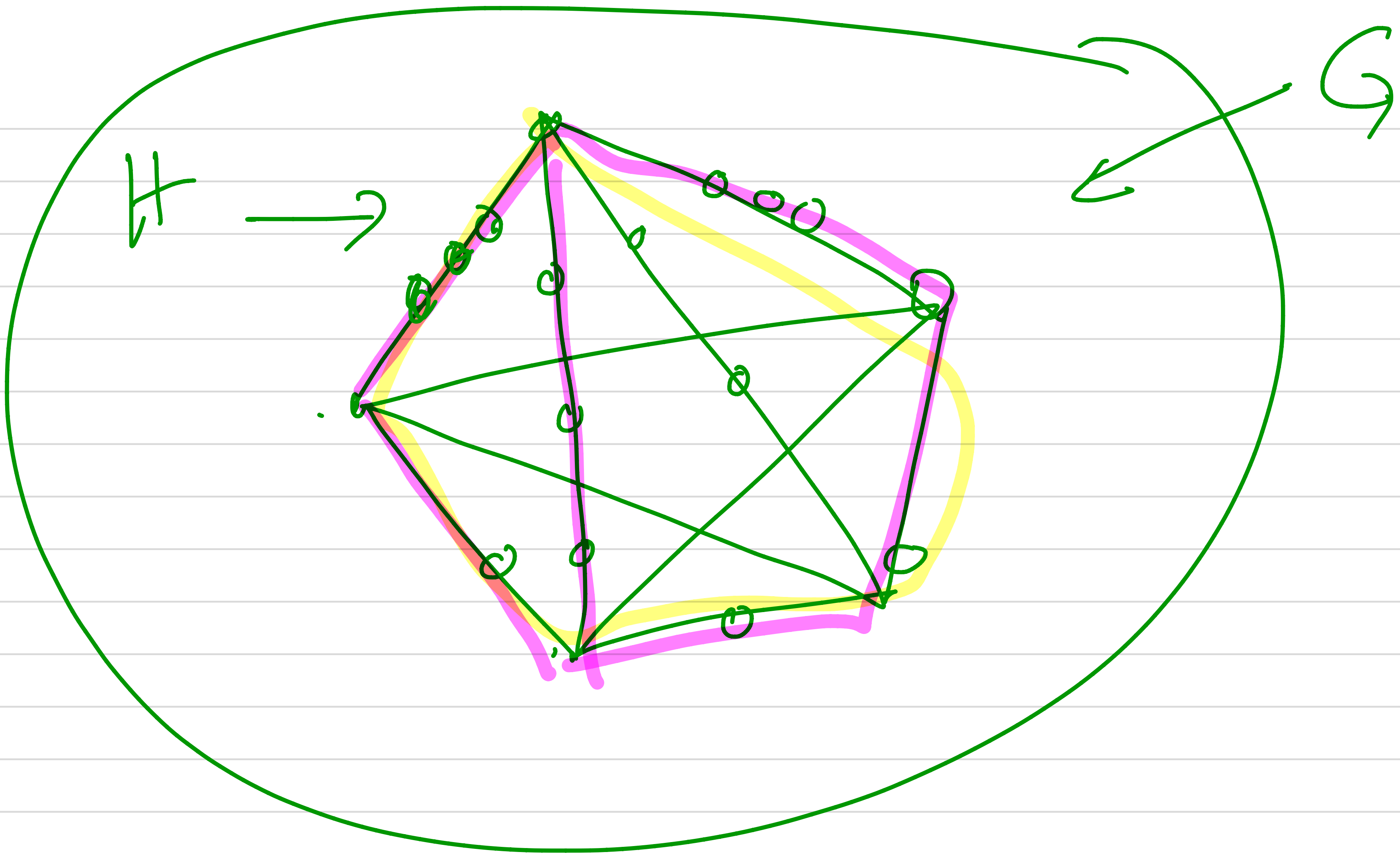


Obs: K^5 and $\boxed{K_{3,3}}$ cannot have plane representations

• K^5: $n = 5$, $m = 10$ to be planar we need
$10 = m \leq 3n - 6 = 15 - 6 = 9$ X
No K^5 is not planar.

• If $K^5 \subseteq G$, then G not planar

• If H is a subdivision of K^5 and $K^5 \subseteq G$, is G planar?



G planar?

If G has a plane embedding then H has a plane embedding

We would get an embedding of K^5 by replacing paths in H by arcs.

Cor 4.2.11

No plane graph can contain a topological minor of K^5 or $K_{3,3}$.

OR

If G contains K^5 or $K_{3,3}$ as a topologial minor, then G is nonplanar.

$\Rightarrow$

Thm (4.4.6) Kuratowski's Thm

G is planar $\Leftrightarrow$ G does not contain K^5 or $K_{3,3}$ as a minor

Logical Structure

- G has no $K_{3,3}$ or K^5 as a minor $\Leftrightarrow$ G has no $K_{3,3}$ or K^5 as a topological minor
- $\Rightarrow$: done.
- $\Leftarrow$: on 3-connected graphs.
    - If G is 3-connected, then $\exists\, e \in G$ s.t. G/e is still 3-connected.
- Any G with no K^5 or $K_{3,3}$ as a top minor and is edge-maximal w.r.t absence of K^5 or $K_{3,3}$ as top minor must be 3-connected.

Lemma 4.4.2

G contains K^5 or $K_{3,3}$ as a minor $\iff$ G contains K^5 or $K_{3,3}$ as a topological minor

<u>Pf</u> $\Leftarrow$ done

$\Rightarrow$: