Fri 20 Oct

- Start Ch5 on Coloring
- Hmwk 6 due today
- Picking a problem topic + some source is on Hmwk 7

Ch 5 Colouring

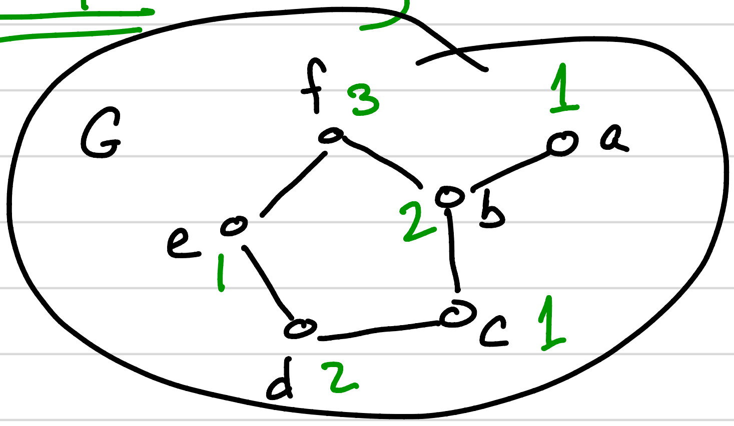
- **def**: A vertex coloring of $G=(V,E)$ is

  $c: V \to S$ s.t. $c(v) \neq c(u)$ if $uv \in E(G)$

  S- colors, $S = \{1,2,\dots,k\} = [k]$

  proper coloring

- Ex:



$S=\{1,2,3\}$ 3-coloring
10-coloring

$\chi(G) = 3$ ← Proof requires w/ demonstrate no 2-coloring.

$\chi(G) \leq 3$

- **def**: A k-coloring of $G=(V,E)$ is a coloring when $S=[k]$

- **def**: The chromatic number of $G=(V,E)$, $\chi(G)$, is the smallest k for which there is a k-coloring of G.

- Color classes induce independent sets of vertices
- G 2-colorable $\Leftrightarrow$ G bipartite
- $G = C^{2k+1}$ $\Rightarrow$ G is not bipartite $\Rightarrow$ $\chi(G) \geq 3$.

$\chi(C^{2k+1}) = 3$, $\chi(P^m) = 2$, $\chi(K^n) = n$, $\chi(K^n \cup K^m) = n$, $n \geq m$

- The difference between:

G is k-chromatic	G is k-colorable
∃ coloring of G w/ k-colors →	
+ ∄ coloring of G w/ k-1 colors	

Thm 5.1.1 Four Color Theorem

Every planar graph is 4-colorable.

Thm 5.1.3

Every triangle-free planar graph is 3-colorable.

Prop 5.1.2 The Five Color Theorem

Every planar graph is 5-colorable.

§ 5.2

C^k

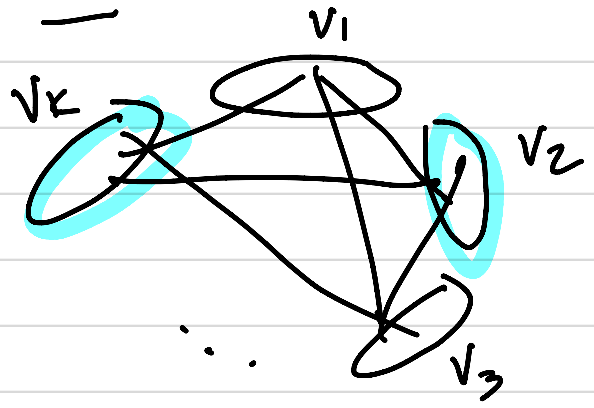
$\chi(C^k) \le \frac{1}{2} + \sqrt{2k + \frac{1}{4}}$

$\approx \sqrt{2k}$

Prop 5.2.1 G has m edges.

Then $\chi(G) \le \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$

Pf: Let G have some k-coloring.



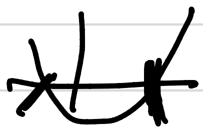
V_i is independent

$\exists$ at least 1 edge betw V_i + V_j, $i \ne j$

$m = \#\text{edge} \ge \binom{k}{2} = \frac{k(k-1)}{2}$

$0 \ge k^2 - k - 2m$

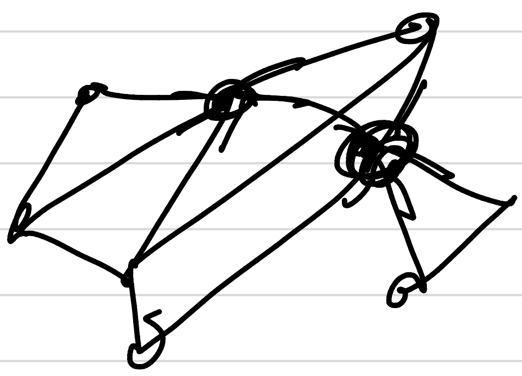
$k = \frac{1}{2} \pm \frac{1}{2}\sqrt{8m+1}$



Embedded Lemma:

G graph w/ $\Delta(G) = \Delta$.

Then $\chi(G) \le \Delta + 1$

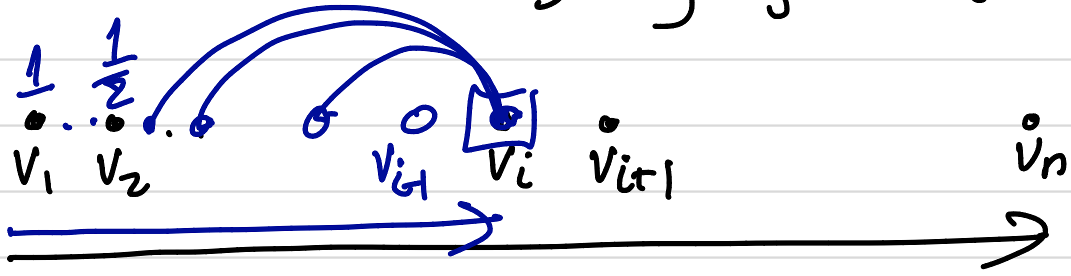


Pf: Apply a greedy coloring algorithm

Given V. Order V arbitrarily $v_1, v_2, v_3, \ldots, v_n$

and colors $\{1, 2, 3, \ldots, \Delta, \Delta+1\} = [\Delta+1]$

- $c(v_1) = 1$
- $\forall i \ge 2$, assign v_i the smallest available color not used by any $v_j \in N(v_i)$ where $j < i$.



Always an available color b/c $d(v_i) \le \Delta$ and even if all $N(v_i)$ has been colored and all got different colors, $[\Delta+1]$ still has at least one available color.

Thm 5.2.4 (Brooks Thm)

If G is connected and not complete and not an odd cycle,

then $\chi(G) \leq \Delta(G)$.

Pf: connected $\Delta \in \{0,1\}$ K^1, K^2

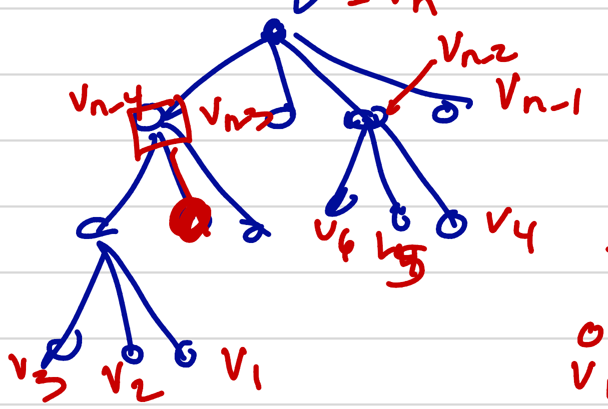
$\Delta = 2$ path or (not odd) cycle
$\chi(G) = 2$

$\Delta \geq 3$

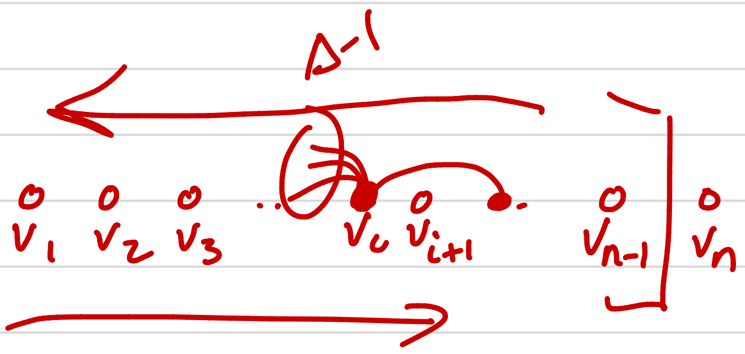
Case 1: $\exists v$ s.t. $d(v) < \Delta$.

Construct spanning tree in G starting at v by breadth first

$v = v_n$



Claim Greedy Alg. using this ordering uses $\leq \Delta$ colors.



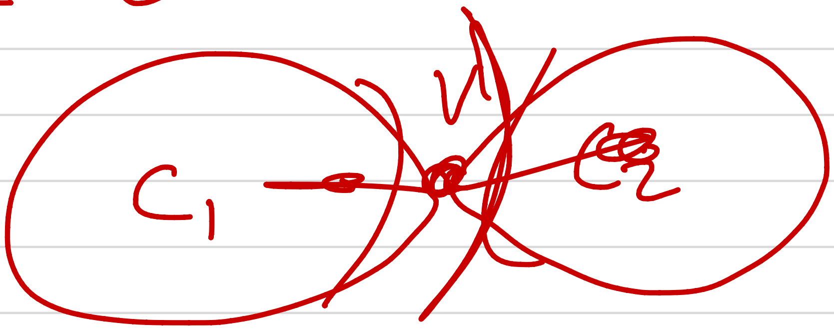
$d(v_n) < \Delta$

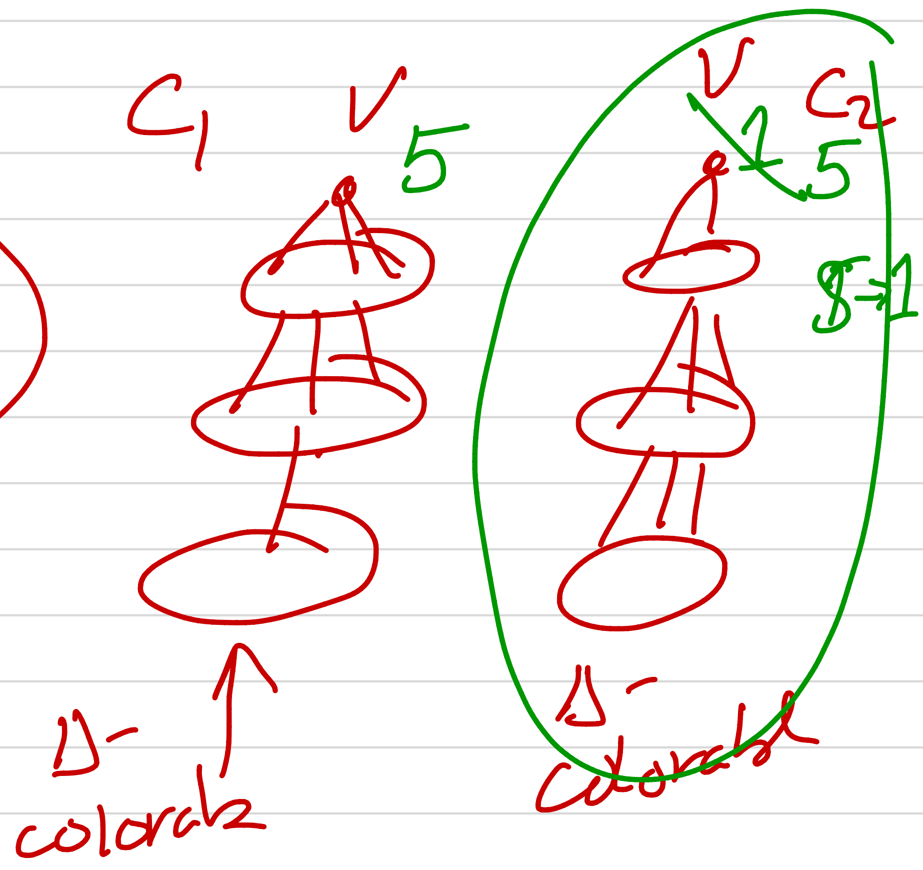
$\forall i \leq n-1$, $\exists v_j$ s.t. $j > i$ and $v_j \in N(v_i)$

So we are guaranteed a color is available for v_i.

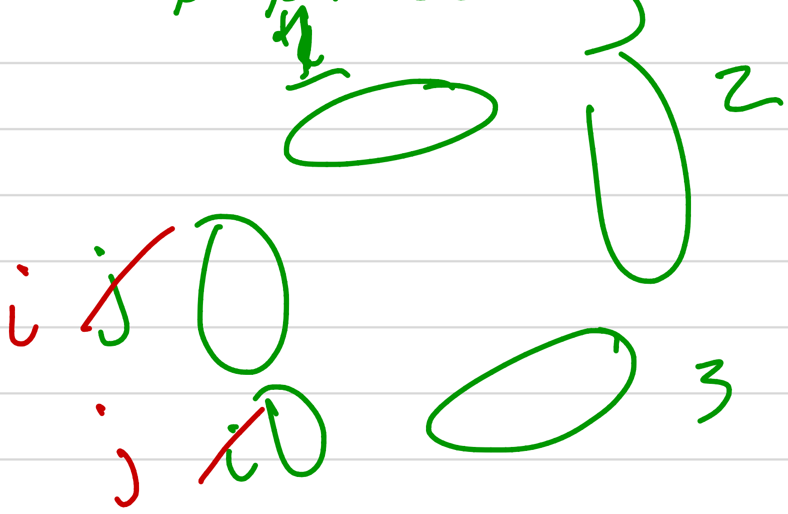
Case 2 : G is reg. of degree Δ.

Subcase (i) $\kappa(G) = 1$





- WOLG we can exchange colors of vertices in 2 distinct color classes and preserve the proper coloring



exchange $c(v)$ in C_2 for color of c in C_1 ~~at~~ in C_2.

$$c : V \to [k] \quad , \sigma \in Sym[k]$$

$$c' = \sigma \circ c$$