Fri 3 Nov

- Hmwk 8 due today
- Agenda
  - Prove Ford-Fulkerson
  - Start Ch7 Extremal Theory
- Two Weeks to Mid 2.
  Should we schedule the Fairbanks Midterm on Thus from 2:30-4:30?
- Monday before Thanksgiving will be a Zoom lecture.

Thm 6.2.2 Ford-Fulkerson

Given $N = (G, s, t, c)$ network. Let f represent a flow on N.

$$\max_{\substack{f \text{ flow on} \\ N}} |f| \overset{\leq}{=} \min_{\substack{S \subseteq V \\ S \text{ a cut in } N.}} \{c(S, \bar{S})\}$$

Pf: • It is sufficient to construct f and find S so that

$$|f| = c(S, \bar{S}).$$

• Our construction implies f is integral.

• Start with $f_0(\vec{e}) := 0$. (integral, valid flow)

• Sppse we have flows $f_0, f_1, \ldots, f_k$ so that

$$|f_i| + 1 \leq |f_{i+1}|$$

• How to construct f_{k+1}?

• def: $W = x_0 \vec{e}_0 x_1 \vec{e}_1 x_2 \ldots x_{\ell-1} \vec{e}_{\ell-1} x_\ell$ a $\boxed{\text{good}}$ walk sv

if $x_0 = s$, $x_\ell = v$ and $\forall\ 1 \leq i \leq \ell - 1$ $c(\vec{e}_i) > f(\vec{e}_i)$

$\forall i$, $\vec{e}_i = (e_i, x_i, x_{i+1})$

define

$$S_k = \{s\} \cup \{v \in V : \exists \text{ a good } s\,v \text{ walk}\}$$

Case 1: $t \in S_k$ (Case 2 $t \notin S_k$)

Pf: $\exists$ a good st path in N.

P: $\overset{s}{x_0} \vec{e}_0 x_1 \vec{e}_1 \ldots \quad x_{\ell-1} \vec{e}_{\ell-1} x_\ell = t$

$$\text{Let } \varepsilon = \min_{\vec{e}_i \in \vec{P}} \{ c(\vec{e}_i) - f(\vec{e}_i) \} > 0 \quad \text{and an integer.}$$

$\mathbb{Z}$ $\mathbb{Z}$

$$f_{k+1}(\vec{e}) = \begin{cases} f_k(\vec{e}) + \varepsilon & \text{if } \vec{e} = \vec{e}_i \\ f_k(\vec{e}) - \varepsilon & \text{if } \vec{e} = \overleftarrow{e}_i \\ f_k(\vec{e}) & \text{if } e \notin E(P) \end{cases}$$

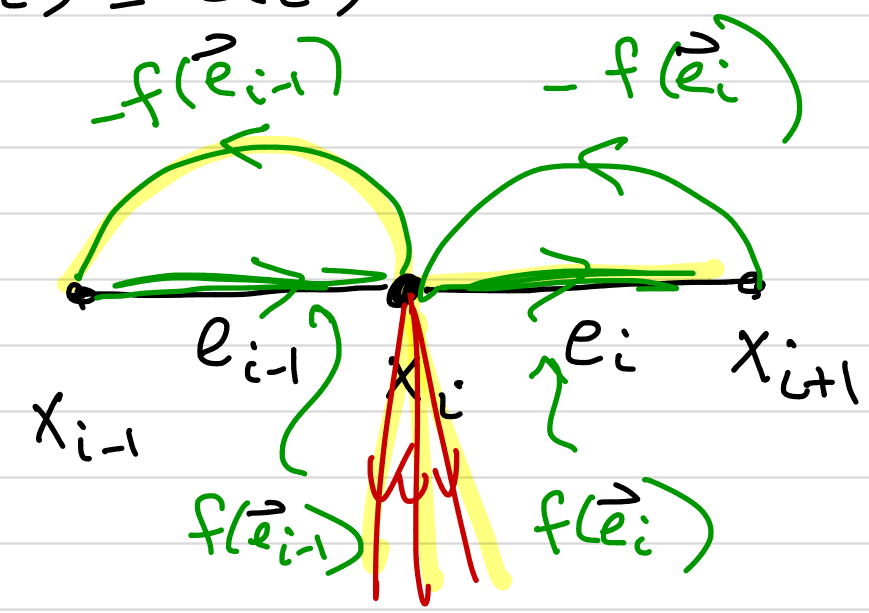
① $f(\vec{e}) = -f(\overleftarrow{e})$

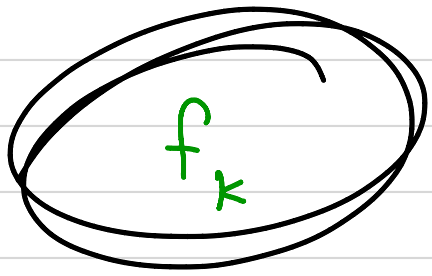
and

$|f_{k+1}| \geq |f_k| + 1$

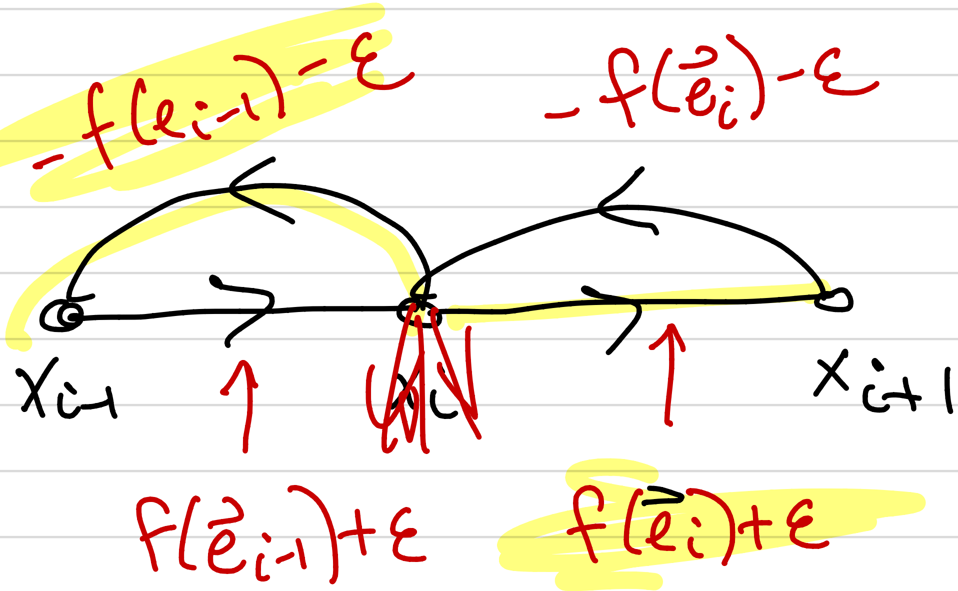
$\boxed{②}$ $f(v, V) = 0$ for $v \in V - \{s, t\}$

③ $\forall \vec{e}$ $f(\vec{e}) \leq c(\vec{e})$





$$\underbrace{\boxed{f_k(x_i, V)} = \text{sum of flows on red edges} + f(e_i) - f(\vec{e}_{i-1})}_{} = 0$$

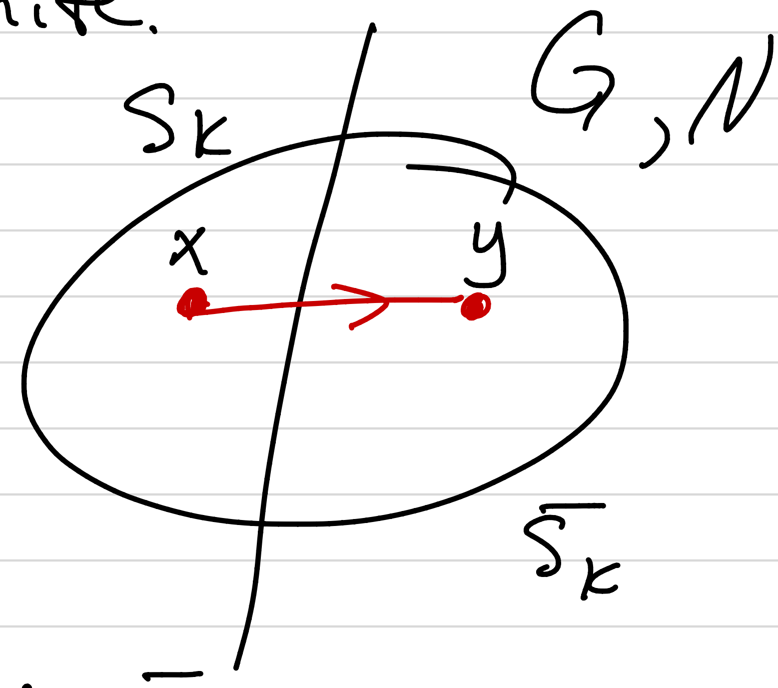


f_{k+1}

$$f_{k+1}(x_i, V) = \text{Sum on red edges} + f_k(\vec{e}_i) + \cancel{\varepsilon} - f_k(\vec{e}_{i-1}) - \cancel{\varepsilon}$$

$= 0$

Obs: This incrementing process must terminate b/c capacities are finite.



Case: $t \notin S_K$

$$|f| = c(S_K, \bar{S}_K)$$

- S_K is a cut b/c $s \in S_K$, $t \in \bar{S}_K$
- $\forall \vec{e} \in E(S_K, \bar{S}_K)$, $c(\vec{e}) = f(\vec{e})$

$e = xy$, $\vec{e} = (e, x, y)$

b/c

otherwise $y \in S_K$.

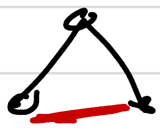
✓

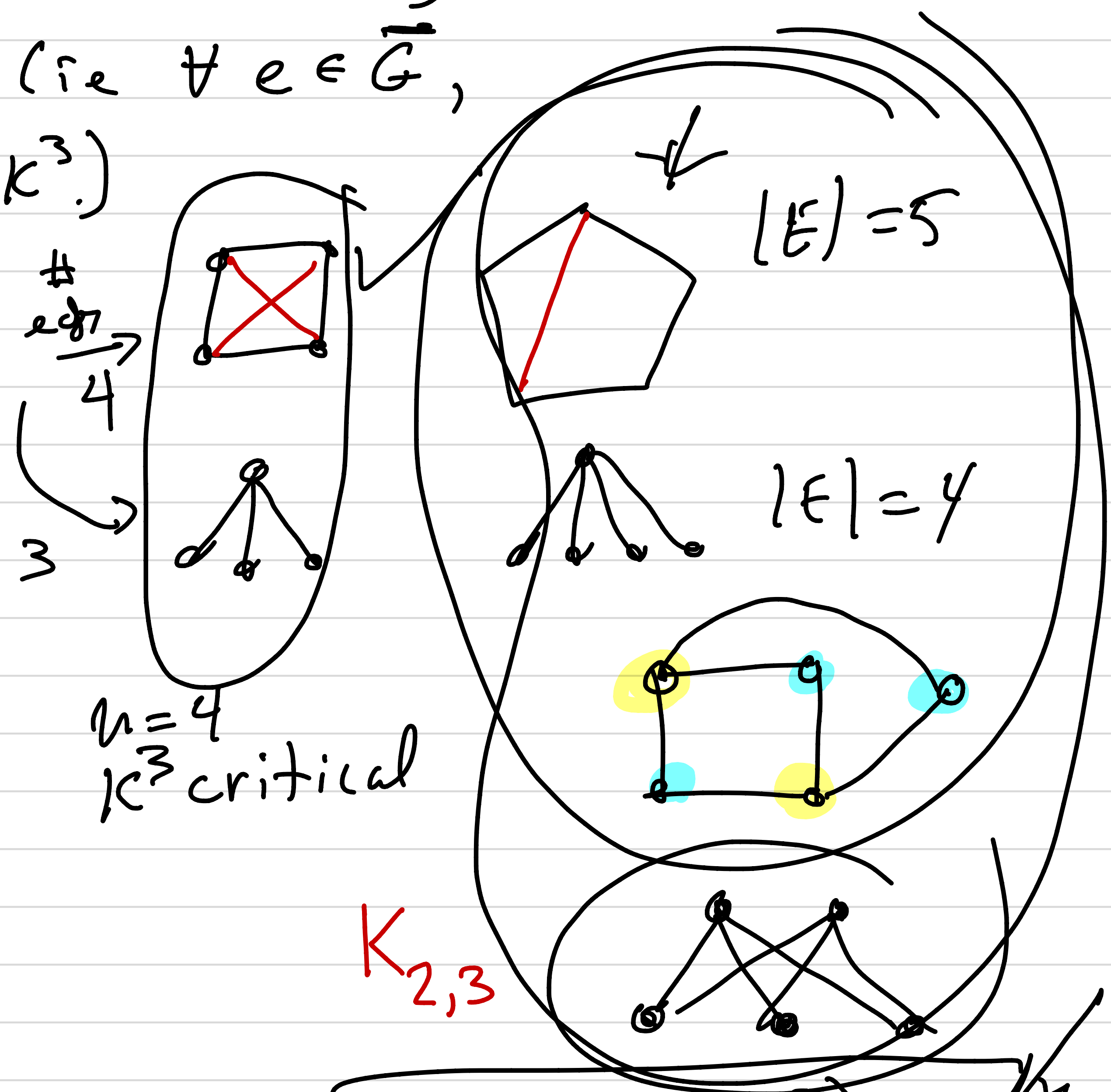
Ch 7 Extremal Graph Theory

Q| Assume # vertices in G is fixed, n.
How many edges can G have and
still not have a K^3?

Obs| Such a graph is necessarily
K^3-critical, (i.e $\forall\, e \in \overline{G}$,
$G + e$ has a K^3)

Ex| ?

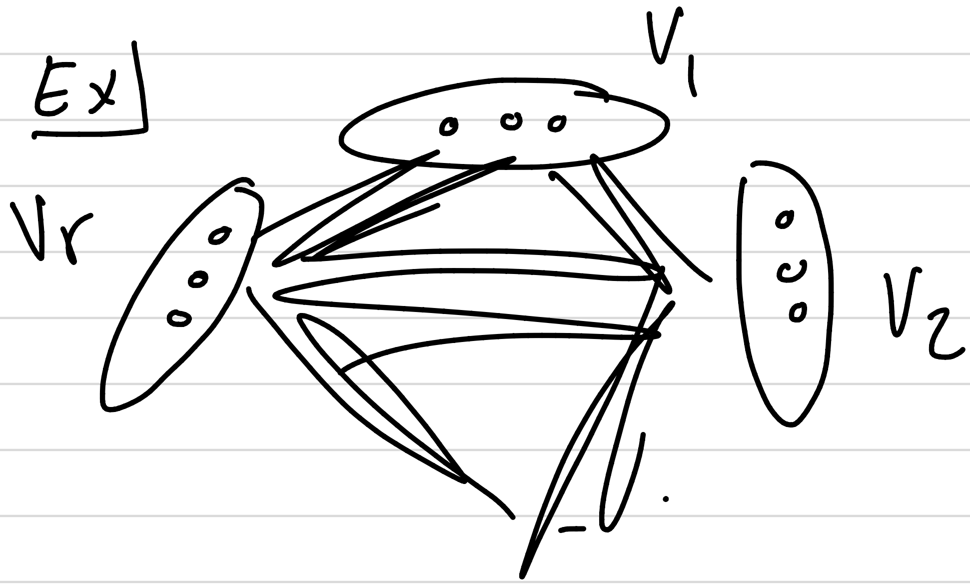




$$\boxed{ex(5, K^3) = 6}$$

How many edges in graph on n vertices that fails to contain a K^r? Sppse $n \geq r$

Ex

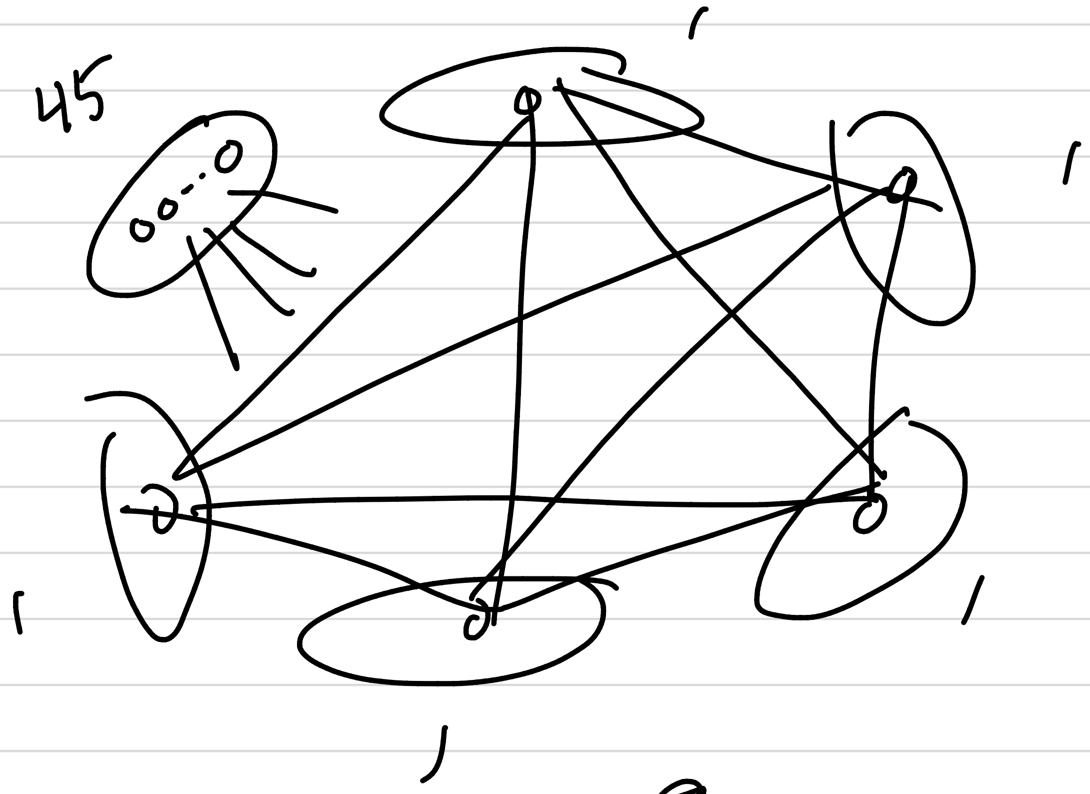


G is $(r-1)$-partite graph.

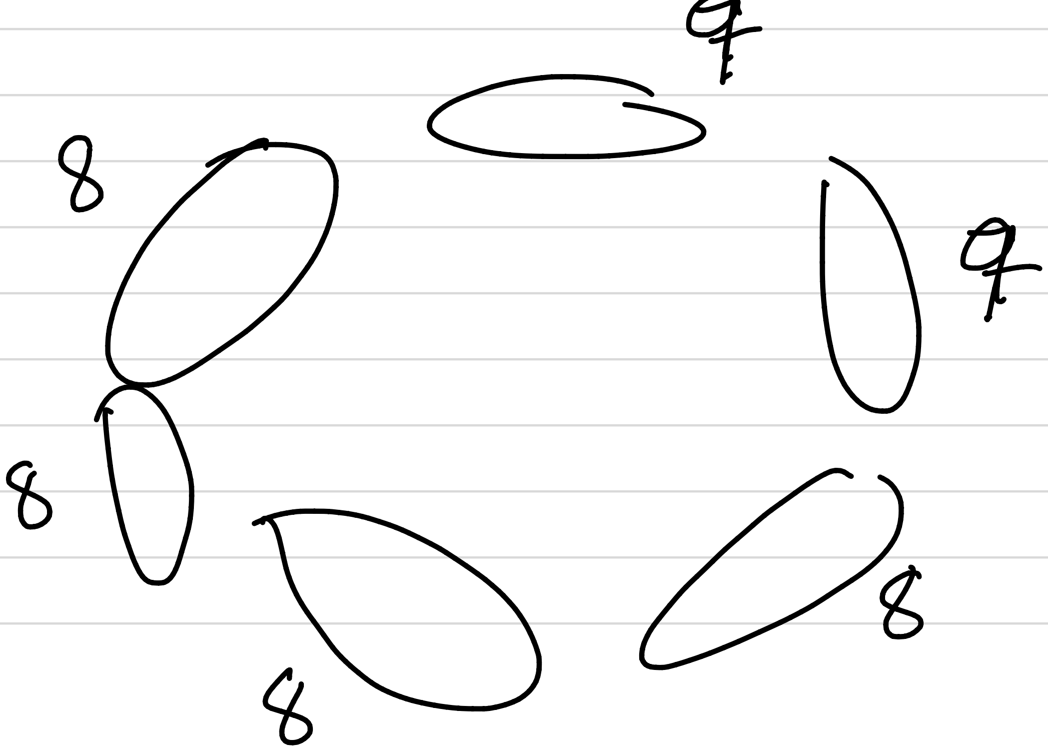
G has no K^r subgraph.

$n = 50 \quad r = 7 \quad$ avoiding K^7

Construct a 6 partite graph



$E(G) = 45 \cdot 5 + \binom{5}{2}$



$$\begin{array}{r} 32 \\ \cancel{44} \\ \hline \cancel{16} \\ 18 \\ \hline 50 \checkmark \end{array}$$

def: $T^r(n)$: an r-partite graph on n vertices such that

the classes are as even as possible.

$\forall$ two class V_i, V_j, $\big|\,|V_i| - |V_j|\,\big| \in \{0,1\}$.

a Turán graph.

Turán's Thm

$$\boxed{ex(n, K^{r+1})} = E(T^r(n)) = t_r(n)$$

P^m

$$t_2(n) = \boxed{\begin{cases} \frac{n^2}{4} & n \text{ even} \\ \frac{n^2-1}{4} & n \text{ odd} \end{cases}}$$

$$= \left\lfloor \frac{n^2}{4} \right\rfloor$$

n even

$$\sum_{1}^{\frac{n}{2}} \cdot \frac{n}{2} = \frac{n}{2} \cdot \frac{n}{2}$$

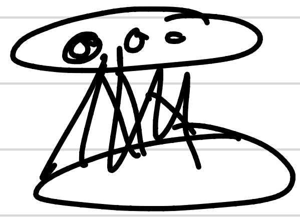
$\frac{n}{2}$

$\frac{n}{2}$

n odd

$$\frac{(n-1)(n+1)}{4} = \frac{n^2-1}{4}$$

$\frac{n-1}{2}$



$\frac{n+1}{2}$

$\left\lfloor \frac{n^2}{4} \right\rfloor + 1$ edss