15 Nov (Wed)

Last of Ramsey Theory notes.

- $R(K^1, K^s) = R(K^s, K^1) = 1$
- $R(K^2, K^s) = R(K^s, K^2) = s.$

- <u>Lemma</u>: $R(K^r, K^s) \le R(K^{r-1}, K^s) + R(K^r, K^{s-1})$

<u>Pf</u>: Let $n = R(K^{r-1}, K^s) + R(K^r, K^{s-1})$.

Given any 2-coloring of K^n, we need to show $\exists$ red K^r or blue K^s.

Let $v \in V(K^n)$. Since $|N(v)| = n-1$, by PHP, $|N_{red}(v)| \ge R(K^{r-1}, K^s)$ or $|N_{blue}(v)| \ge R(K^r, K^{s-1})$.

If $|N_{red}(v)| \ge R(K^{r-1}, K^s)$, then there is a red K^r or a blue K^s in $N_{red}(v)$.

If $|N_{blue}(v)| \ge R(K^r, K^{s-1})$, then there is a red K^r or a blue K^s in $N_{blue}(v)$.

- Find a bound for $R(K^3, K^3)$

Thm: For every $t \geq 3$, $R(K^t, K^t) = R(t) > \lfloor 2^{t/2} \rfloor$.

$(R(3) \geq 2.8)$

Pf: Strategy: • Demonstrate the existence of a graph G such that $K^t \notin G$ and $K^t \notin \overline{G}$.
• Straight counting argument.

Let $n = \lfloor 2^{t/2} \rfloor$. Let $V = \{1, 2, \ldots, n\}$, labeled vertices.

So there exist $2^{\binom{n}{2}}$ distinct labeled graphs on V

and $\binom{n}{t}$ distinct subsets $S \subseteq V$ where $|S| = t$.

Given a particular t-subset S of V, there exist $2^{\left(\binom{n}{2} - \binom{t}{2}\right)}$

graphs such that $G[S] = K^t$.

Let M represent the number of graphs on V that contain a subgraph isomorphic to K^t.

So $M \leq \binom{n}{t} 2^{\binom{n}{2} - \binom{t}{2}} < \frac{n^t}{t!} 2^{\binom{n}{2} - \binom{t}{2}}$

$$\binom{n}{t} = \frac{n!}{t!(n-t)!} = \frac{n(n-1)(n-2)\cdots(n-t+1)}{t!} < \frac{n^t}{t!}$$

Now, $n^t \leq \left(2^{t/2}\right)^t = 2^{t^2/2} = 2^{\left(\frac{t^2}{2} - \frac{t}{2} + \frac{t}{2}\right)}$

$$= 2^{\binom{t}{2}} \cdot 2^{\frac{t}{2}}$$
$$< 2^{\binom{t}{2}} \cdot \frac{1}{2} t!$$

for $t \geq 3$

$\frac{1}{2} t! > 2^{t/2}$.

So $M < \frac{1}{2} \cdot 2^{\binom{n}{2}}$. So $2M < 2^{\binom{n}{2}}$.

So we have shown that deleting all labeled graphs with $K^t \subseteq G$ and all so that $K^t \subseteq \overline{G}$ still leaves at least 1 graph.