Mon 23 Oct

- Hmwk #7 posted.
  (Added directions Sun. morning re:
  when you do or don't need a justification)

- Today
  - Finish 5.2.
  - Mycielski's Construction. See Daily Log
  - Start 5.3

Recall:

- G is k-colorable means it is possible to assign 1 of k colors to each vertex s.t. adj. vertices get different colors.

- $\chi(G) = k$ means the lowest k-value s.t.
  chromatic #
  "G is k-colorable" is true.

- If $c : V \to [k]$ is a k-coloring of G
  then
  $i \in \{1,2,\dots,k\}$
  the i^{th} color class is $V' \subseteq V$ s.t.

  c assigns the color i to all $v \in V'$.
  or i^{th} color class is $c^{-1}(i)$
  color classes are independent sets of vertices

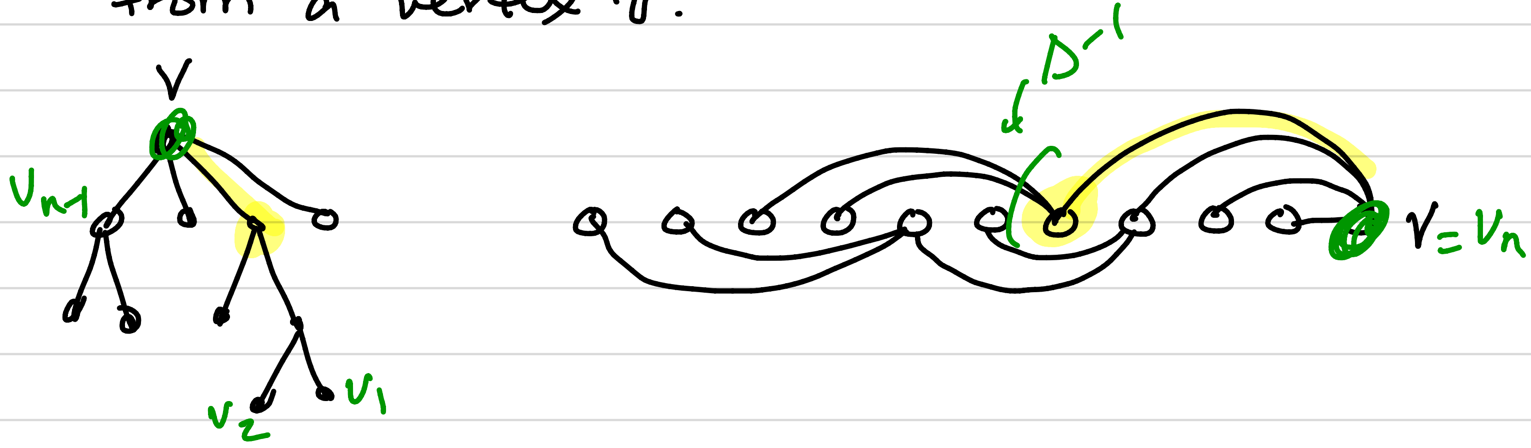
- Brooks' Thm states
  If G is connected and not C^{2k+1} and not K^n, then
  $$\chi(G) \leq \Delta(G).$$

Brooks' Thm

G connected, not C^{2k+1}, not K^n, then

$$\chi(G) \le \Delta(G)$$

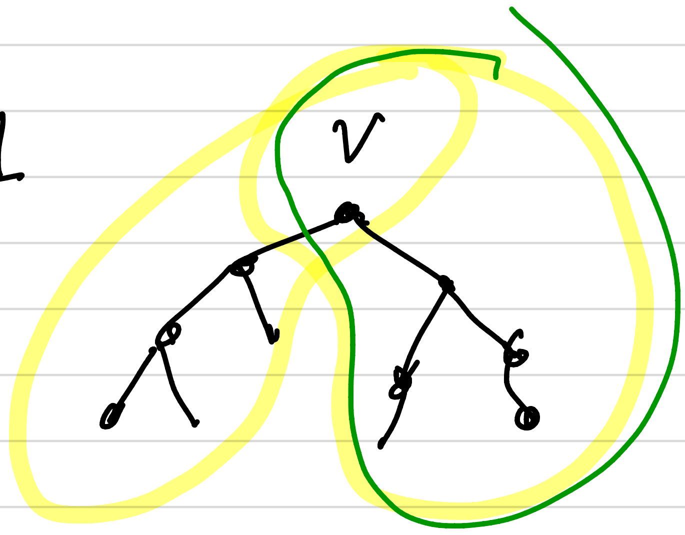
Pf: Strategically grow a spanning tree from a vertex v.



Case 1 If $\exists v \in V$ s.t. $d(v) < \Delta(G)$

Case 2 G is Δ-regular

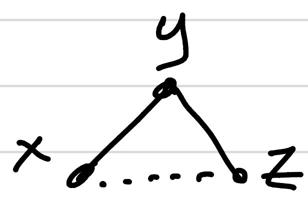
Subcase 2.1 $K(G) = 1$



K_2
$\Delta = 2$

Subcase 2.1 $K(G) \geq 2$

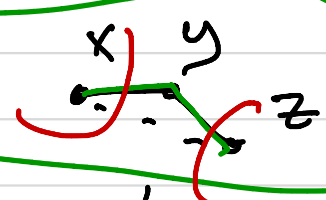
Goal: $\{x, y, z\} \subseteq V(G)$ s.t.



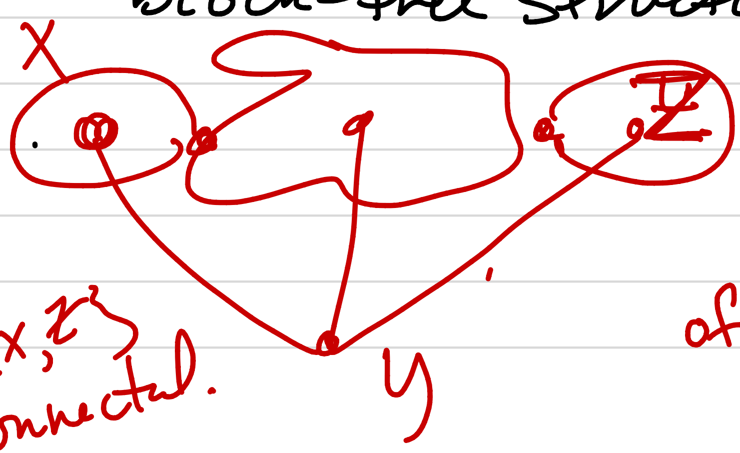
and $G - \{x, z\}$ is connected.

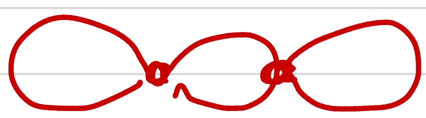
Δ-reg.
$\Delta \geq 3$
G is 2-conn.
$\chi(G) = 1$

If $G - x$ is 2-connected, done

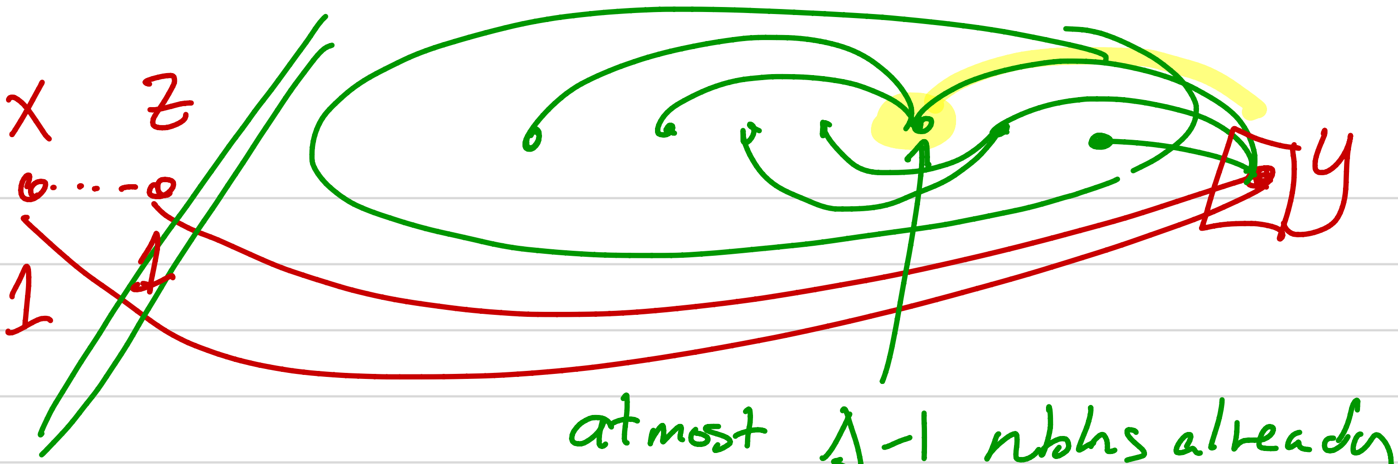


So $G-y$ has connectivity 1. So $G-y$ has a block-tree structure. + thus has end-blocks



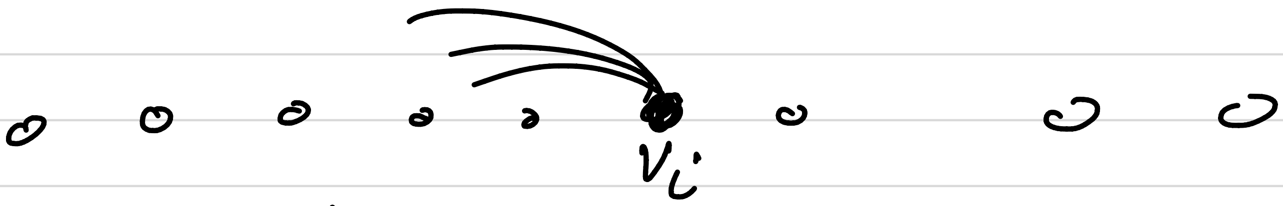


y has a neighbor in every endblock of which there are at least 2.



atmost $\Delta-1$ nbhs already colored.
So 1 color is available.

At y, also only $\Delta-1$ colors have been used on $N(y)$.



Δ-colors

Two Other Results

Lemma 5.2.3

If $\chi(G)=k$, then $\exists\, H\subseteq G$ s.t. $\delta(H)\geq k-1$

and $\chi(H)=k$.

Pf: G graph, $\chi(G)=k$.

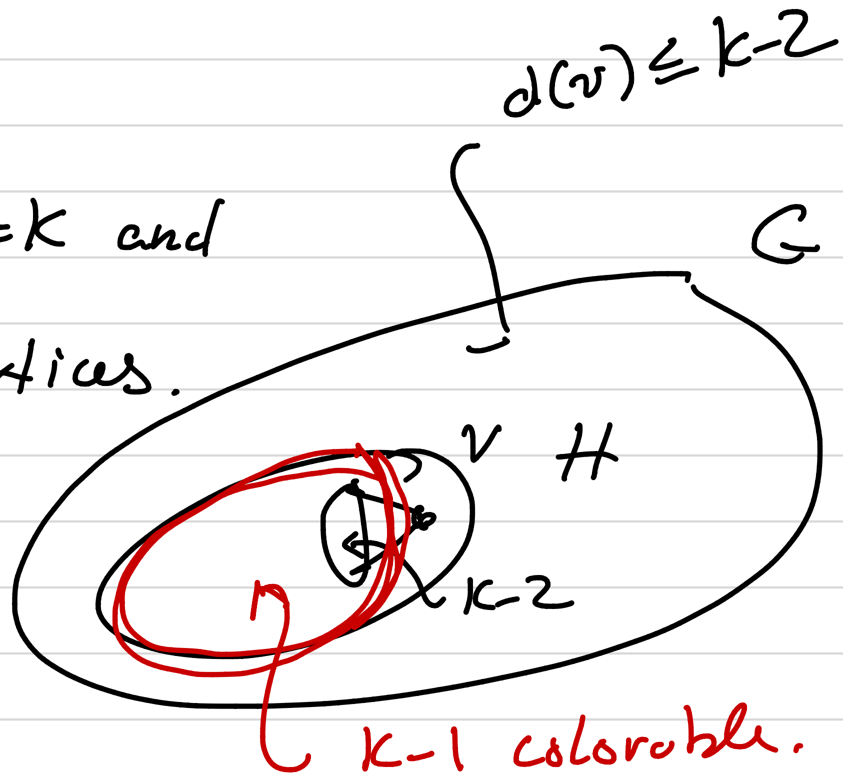
Choose $H\subseteq G$ w/ $\chi(H)=k$ and

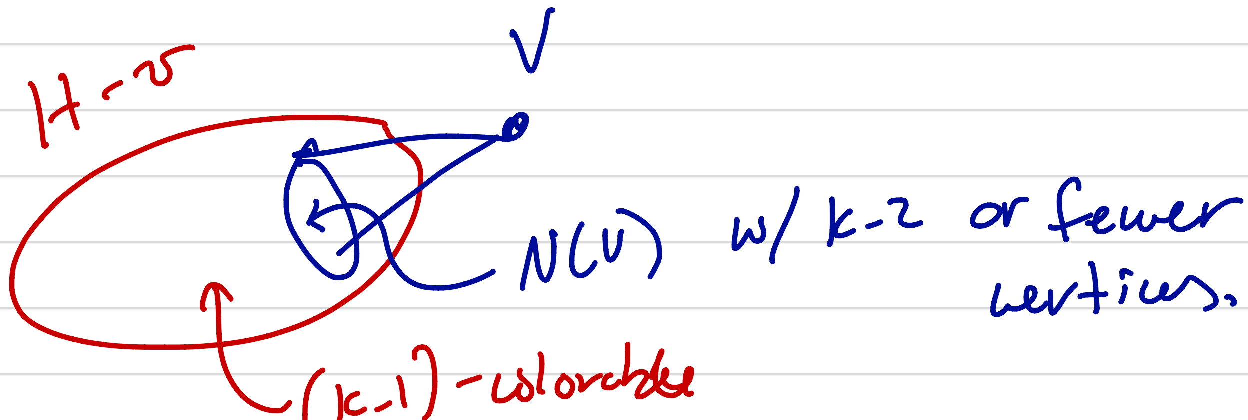
H is minimum wrt # vertices.

Claim: $\delta(H)\geq k-1$.

What if $\delta(H)\leq k-2$?

$\chi(H-v)\leq k-1$





So color v w/ one of remaining colors from $[k-1]$,

So H is $(k-1)$-colorable $\Rightarrow\Leftarrow$ $\chi(H)=k$.

Thm 5.2.5 (Erdős)

$\forall\, k\in\mathbb{Z}^+$, $\exists\, G$ s.t. $g(G)>k \wedge \chi(G)>k$.

$k=3$, $\exists\, G$ s.t. $g(G)>3 \wedge \chi(G)>k$.

Mycielski's Construction

To show $\exists$ graphs G such that

- G is triangle free
- $\chi(G) \geq k$ for any $k \in \mathbb{Z}^+$.

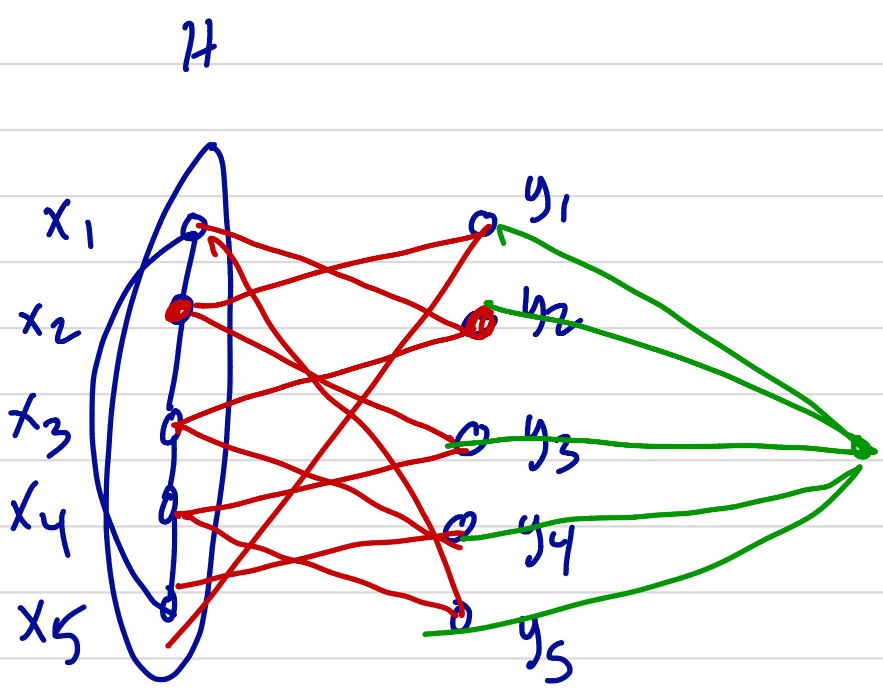
<u>Construction</u> Given $H = (V, E)$, construct G via

$V(G) = V \cup V' \cup \{z\}$ where $|V'| = |V|$.

$(V = \{x_1, x_2, \dots, x_n\},\ V' = \{y_1, y_2, \dots, y_n\})$

$E(G) = E(H) \cup \{x_j y_i : x_j x_i \in E(H)\} \cup \{z y_i : y_i \in V'\}$

$H = C^5 \qquad V = \{x_1, x_2, x_3, x_4, x_5\}$



$G \qquad \Delta$-free

$\chi(G) = 4$

Observation

① If H is Δ-free, then G is Δ-free. $\boxed{\text{You}}$

② If H is k-colorable, then G is $(k+1)$-colorable. $\boxed{\text{You}}$

③ If $\chi(H) = k$, then $\chi(G) > k$. $\boxed{\text{Us}}$

$M(G) = \tilde{G} \qquad \tilde{G}\ \Delta$ free and $\chi(\tilde{G}) = 5$