# Mon 6 Nov

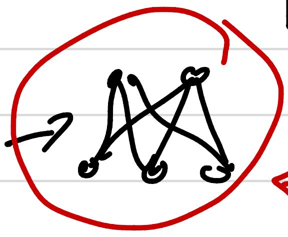
- Hmwk #9 due Fri
- Hmwk #8 solns posted
- Midterm 2 in $\approx$ 2 weeks.

- <u>Agenda</u> Turán's Thm

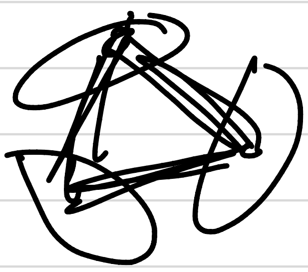
## <u>Ch 7</u>

- $ex(n, H)$ = max # edges on an $n$-vertex graph, $G$, s.t. $H \nsubseteq G$.

- $ex(n, K^3) = |E($complete balanced or nearly balanced bipartite graph$)|$
$$= \left\lfloor \frac{n^2}{4} \right\rfloor$$

$n=5$  

$K_{2,3} = T^2(5)$



- $T^r(n)$ = complete $r$-partite graph on $n$ vertices s.t. classes are as balanced as possible.

  Turan graphs.

- $t_r(n) = |E(T^r(n))|$

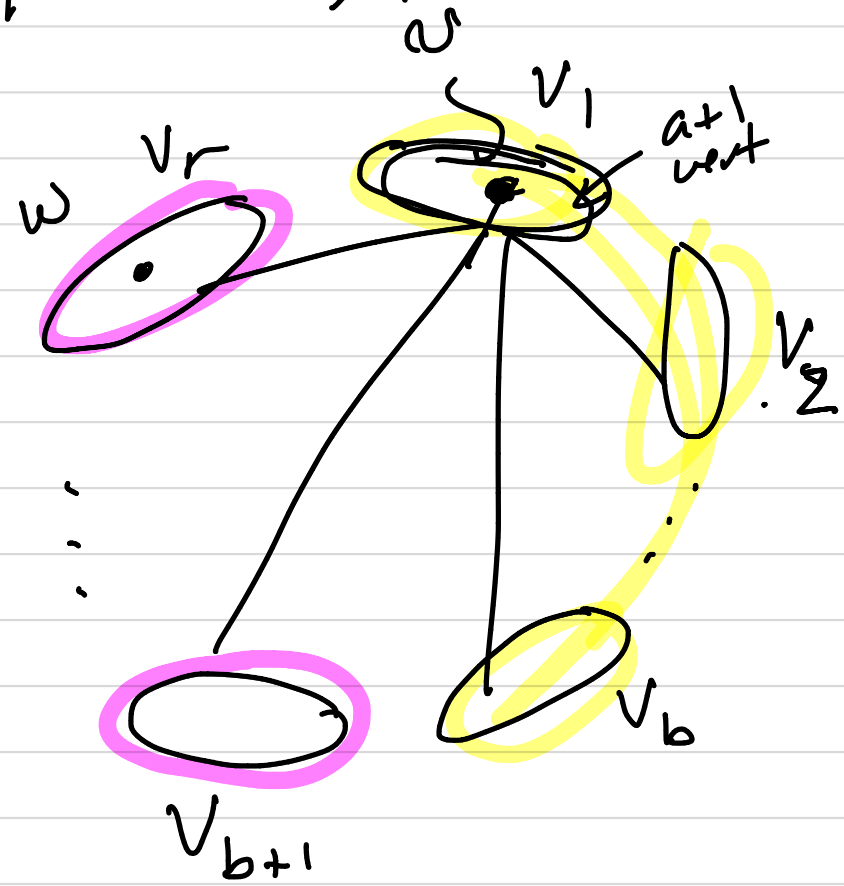
- Turan's Thm $\quad ex(n, K^r) = t_{r-1}(n)$

• The graph $G$ is $H$-free means $H \not\subseteq G$.

• Count $|E(T^r(n))|$ $\quad n = ar + b \quad 0 \le b < r$

b of $V_i$'s have $a+1$ vertices
$r-b$ have $a$ vertices.

$T^r(n)$




$$|E(T^r(n))| = \frac{1}{2}\left(\text{deg sum}\right)$$

$$= \frac{1}{2}\left( \boxed{\underbrace{b}_{\substack{\text{diff}\\\text{yellow}\\\text{class}}}\underbrace{(n-(a+1))}_{d(v)}\underbrace{(a+1)}_{\substack{\#\text{per}\\\text{class}}}} + \underbrace{(n-a)}_{d(w)}\underbrace{(a)}_{\substack{\#\text{per}\\\text{class}}}\underbrace{(r-b)}_{\substack{\#\\\text{classes}}} \right)$$

$$= \frac{n^2}{2}\left(1 - \frac{1}{r}\right) - \frac{b(r-b)}{2r}$$

$$\le \frac{1}{2} n^2 \left(\frac{r-1}{r}\right)$$

Turán's Thm (Sppse $r \geq 2$)

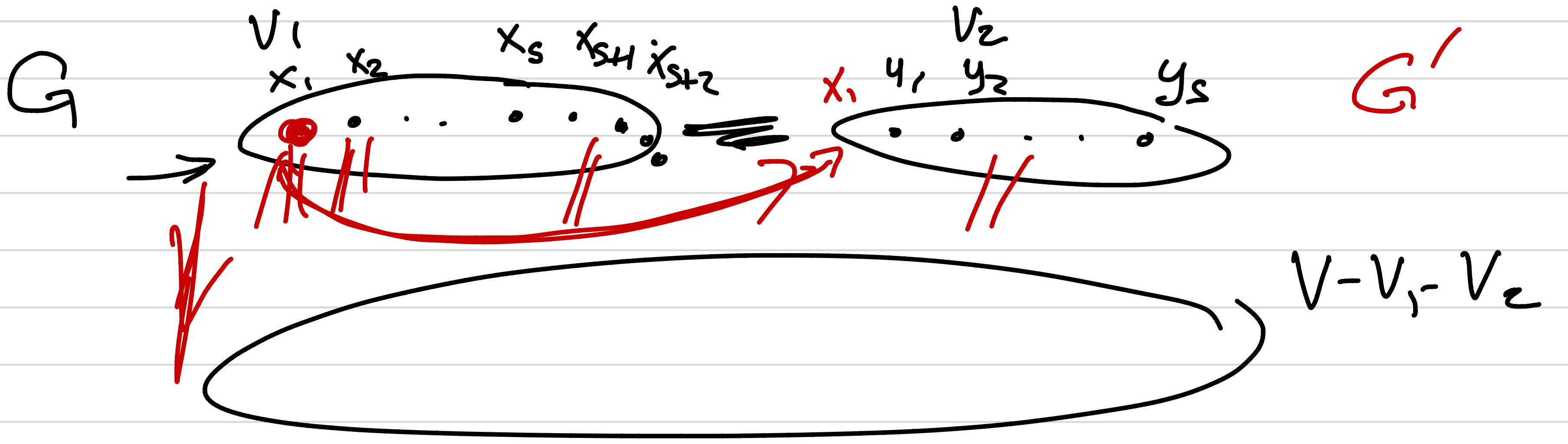
- $ex(n, K^r) = t_{r-1}(n)$
- Moreover, if $K^r \nsubseteq G$ and $|E(G)| = ex(n, K^r)$, then $G \cong T^{r-1}(n)$

alt: Extremal graph is unique.

Pf: ($2^{nd}$ proof + embedded argument)

- Strategy is to show that if $G$ is $K^r$-free w/ max # edges then $G \cong T^{r-1}(n)$.
- $G$ is (complete) $(r-1)$-partite or not.
- Among all ~~If G is~~ complete $(r-1)$-partite graphs on $n$ vertices, then ~~a~~ one w/ max # of edges has classes as equal in cardinality as possible.

Sppse classes $V_1$ & $V_2$ have $|V_1| = |V_2| + 2 + \ell = \Delta + 2 + \ell$




Counting the change in # edges when $x_1$ moves from $V_1$ to $V_2$.

Only edges that are changed are those between $V_1$ & $V_2$

before: $|E(G[V_1 \cup V_2])| = \Delta(\Delta + 2 + \ell)$

after $|E(G'[V_1 \cup V_2])| = (\Delta + 1)(\Delta + 1 + \ell)$

x-moved

$= \Delta(\Delta + 1 + \ell) + \Delta + 1 + \ell$

$= \Delta(\Delta + 2 + \ell) + 1 + \ell$ ✓

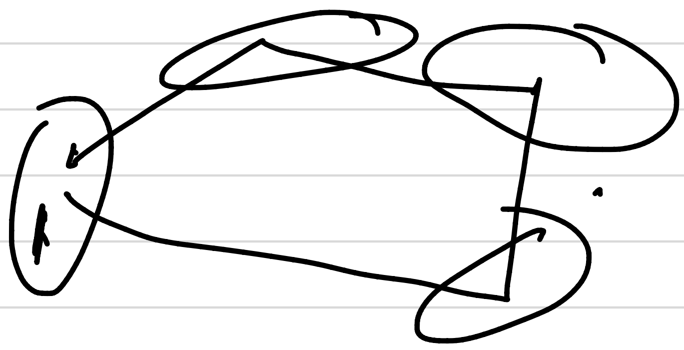
If $G$ is $(r-1)$ partite & complete, it must be as balanced as possible..

- If $G$ is multipartite & complete, why it must have $(r-1)$ classes?

  If $r$, $K^r \subseteq G$

  If $(r-2)$ then fewer edges.

$r-2$



- What if $G$ is not a complete $(r-1)$-partite graph.?

  Claim $G$ does not have a maximum # edges.
  (we show by switching structures we can add more edges)

  Obs: If $G$ is not a complete multipartite graph, then the relation $R$

  called $xRy$ if $xy \notin E(G)$

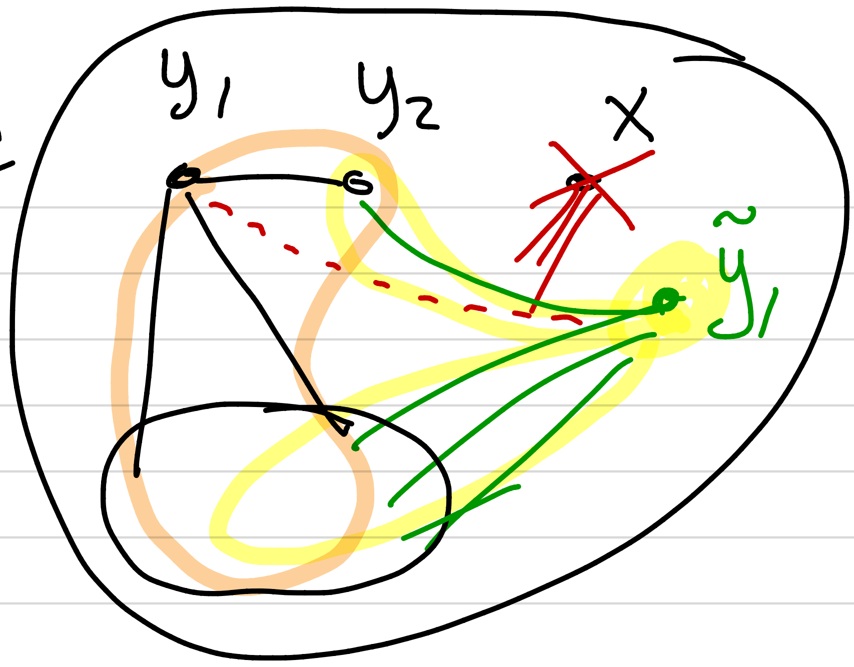
  is not an equivalence relation.

  So $R$ not transitive

  So $\exists\ y_1, x, y_2$ s.t. $xy_1, xy_2 \notin E(G)$

  but $y_1 y_2 \in E(G)$

Case 1



G

next

Case 1: $d(y_1) > d(x)$

Make new graph $\tilde{G}$ by deleting $x$ & duplicating $y_1$. $|E(\tilde{G})| > |E(G)|$

- $K^r \notin \tilde{G}$. If $K^r \subseteq \tilde{G}$, then $\tilde{y}_1 \in V(K^r)$ which implies $K^r \subseteq G$, since $\tilde{y}_1 y_1 \notin E(\tilde{G})$

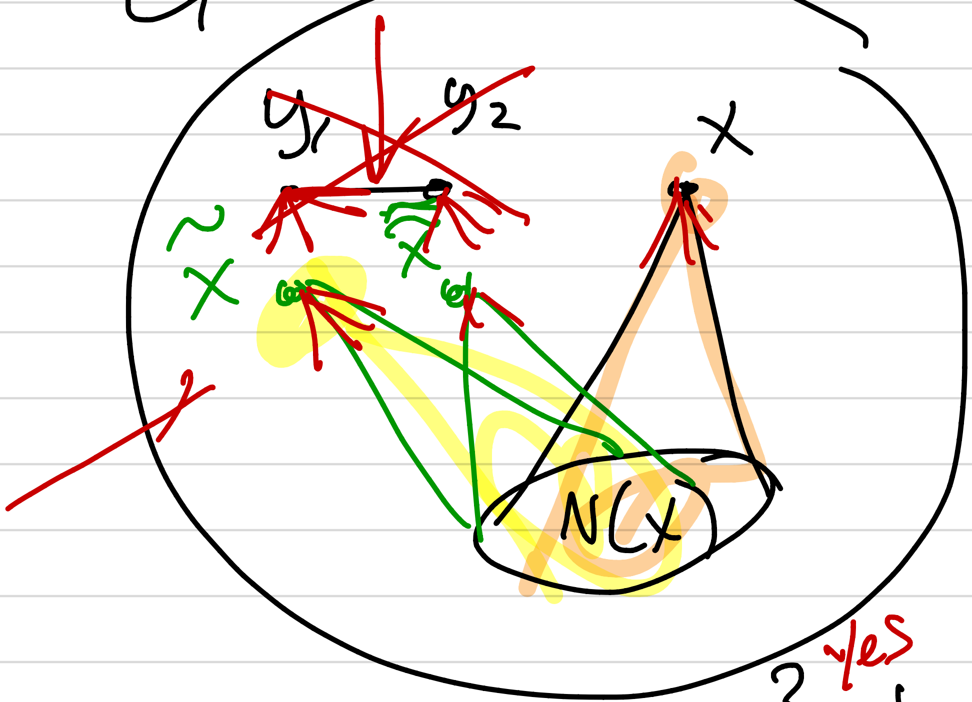
Case 2: $d(y_2) > d(x)$ ✓

Case 3: $d(x) \geq d(y_1)$ and $d(x) \geq d(y_2)$

Make new graph $\tilde{G}$ by deleting $y_1$ & $y_2$ & duplicating $x$ twice

$\tilde{G}$

G



- $|E(\tilde{G})| \overset{?}{>} |E(G)|$ yes

- No $K^r$ in $\tilde{G}$.

G is not complete multi-partite w/ (r-1)-classes on n vertices