Wed 25 Oct

- Hmwk due Fri
- Stuff is posted
- Still working on grading old hmwk

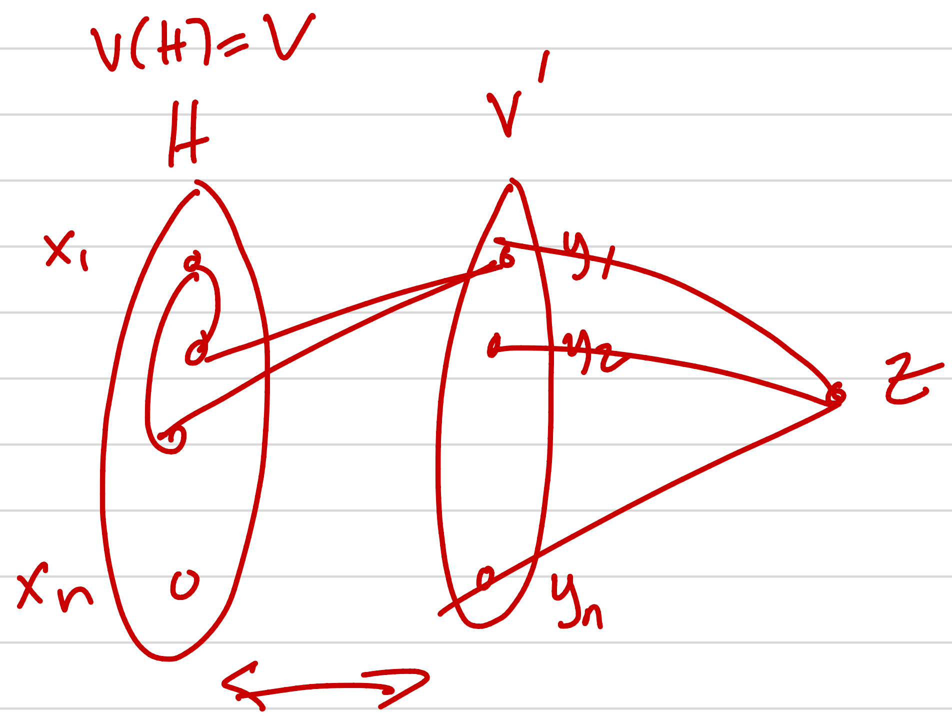
Recall Mycielski's Construction

Construction Given $H=(V,E)$, construct G via

$$V(G) = V \cup V' \cup \{z\} \quad \text{where } |V'|=|V|.$$

$$(V=\{x_1,x_2,\dots,x_n\},\ V'=\{y_1,y_2,\dots,y_n\})$$

$$E(G) = E(H) \cup \{x_j y_i : x_j x_i \in E(H)\} \cup \{z y_i : y_i \in V'\}$$



H: · Δ-free
  · $\chi(H)=k$

$\Downarrow$

G: · Δ-free
  · $\chi(G)=k+1$

$N(y_i) = N(x_i$

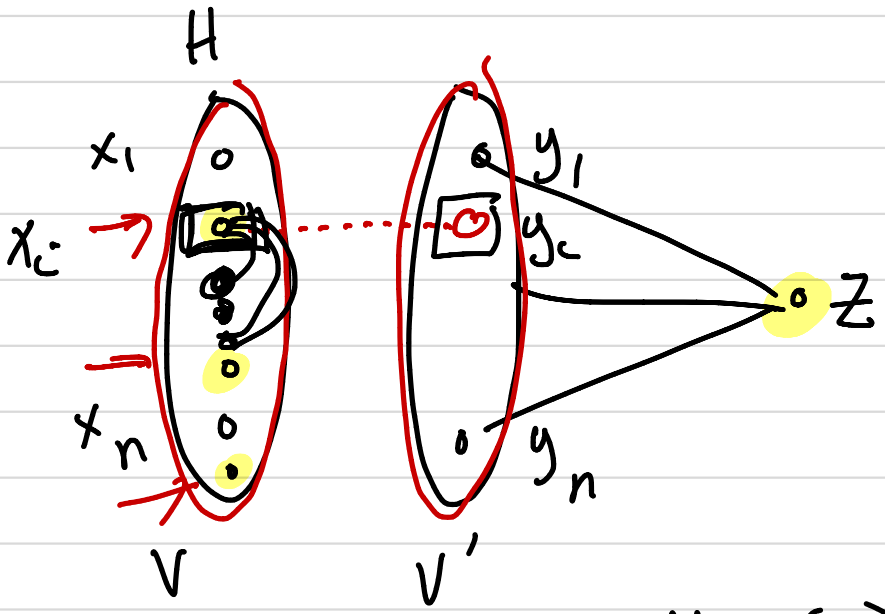
Show: If $\chi(H)=k$, then G is not k-colorable.

Pf: · by contradiction

· If c is a k-coloring of G then we could find a $(k-1)$-coloring of H $\Rightarrow\Leftarrow$

c is a k-coloring of G.

WOLG sppse $c(z)=k$



· $c(z)=k$.

· $\forall i \quad c(y_i) \in \{1,2,\ldots,k-1\}$

· $\boxed{\text{Let } S=\{x_i \in H : c(x_i)=k\}}$

(If $S=\emptyset$, then H was $(k-1)$-colored by c)

New $(k-1)$-coloring of H merely requires re-coloring vertices in S. Do this by:

$\forall x_i \in S, \quad \tilde{c}(x_i) := c(y_i) \in [k-1]$.

$\tilde{c} : V(H) \to [k-1]$ def

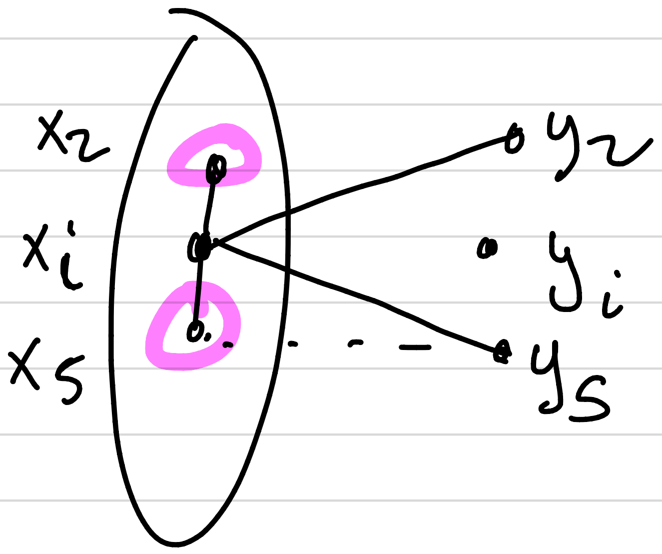
$$\tilde{c}(x_i) = \begin{cases} c(x_i) & x_i \notin S \\ c(y_i) & x_i \in S \end{cases}$$

· Still need to show adjacent have different colors

· ~~Need to check: $\forall x_i \in S$, $\forall x_j \in N_H(x_i)$,~~ $\tilde{c}(x_i) \neq c(x_j)$.

This is true b/c $c(y_i) \neq c(x_j) \;\forall\; x_j \in N_H(x_i)$

$N_H(y_i) = N_H(x_i)$

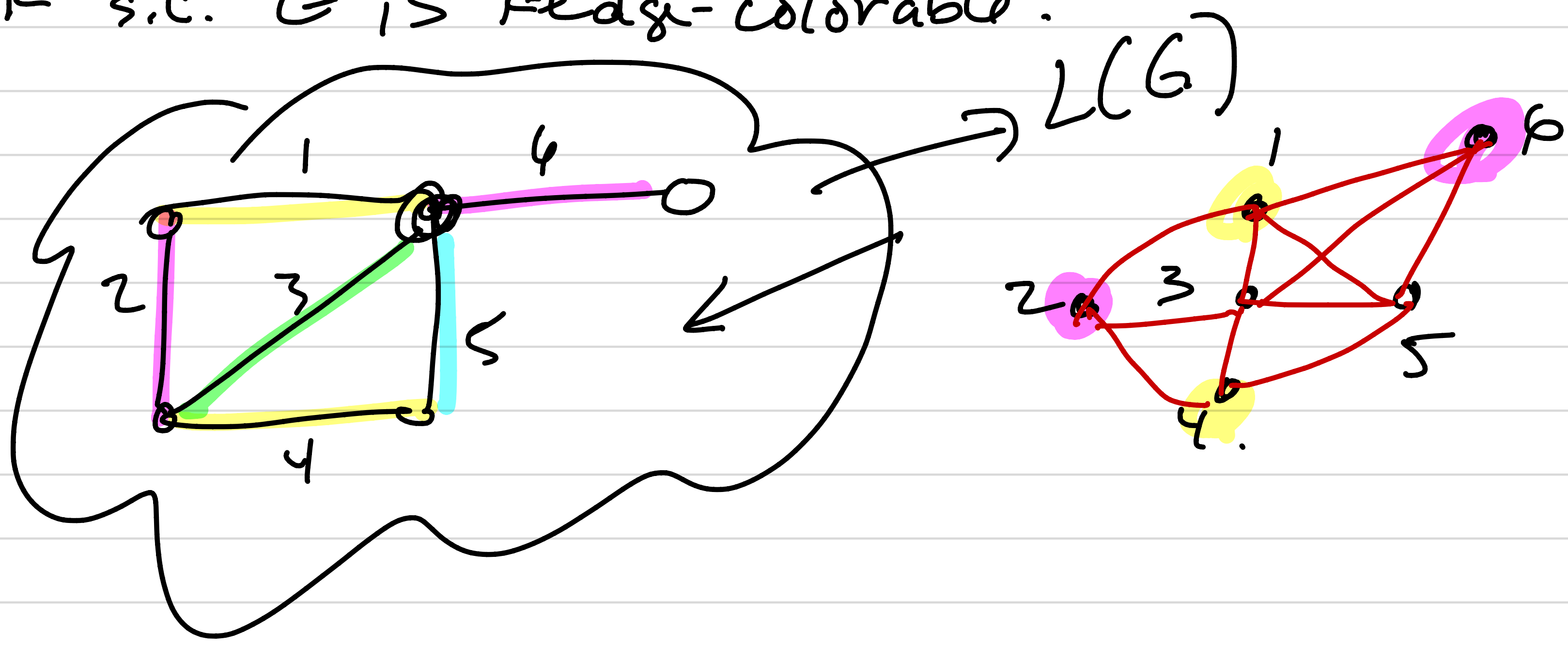


§ 5.3 Edge-Coloring

- <u>def</u>: A k-edge-coloring of $G=(V,E)$ is $c: E \to [k]$ such that if $e_1, e_2 \in E$ share a vertex then $c(e_1) \neq c(e_2)$.

- <u>def</u>: G is k-edge-colorable if $\exists$ k-edge-coloring of G.

- <u>def</u>: The edge-chromatic number of G, denoted $\chi'(G)$, is the smallest k s.t. G is k-edge-colorable.



- Observations
  - $\Delta(G) \leq \chi'(G)$
  - $\chi'(G) = \chi(L(G))$

Prop 5.3.1

If G is bipartite, then $\chi'(G) = \Delta(G)$

Pf: Induction $|E(G)|$.
Sppse G bipartite

Base: $|E(G)| = 0, 1, 2$

Sppse $\chi'(G) = \Delta(G)$ provided G has bipartite & fewer than m edges.

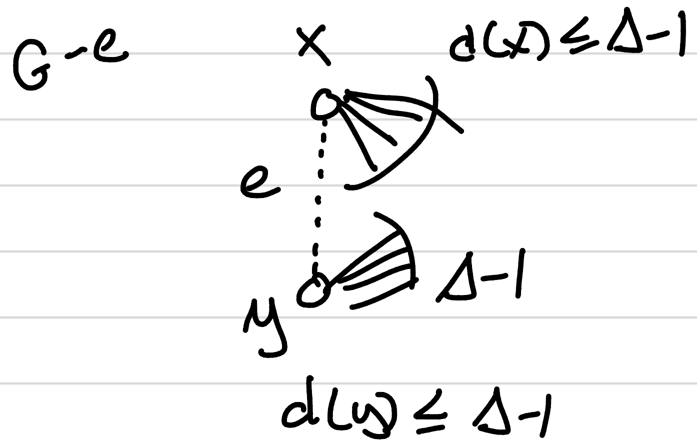
Let G is bipartite on m edges. Let $\Delta = \Delta(G)$

Nts. $\exists$ a Δ-coloring of G.

Let $e = xy \in E(G)$.

Observe by induction $\exists$ a Δ-coloring of $G-e$. ($\Delta(G-e) \le \Delta(G)$)

Say c is a Δ-coloring of $G-e$



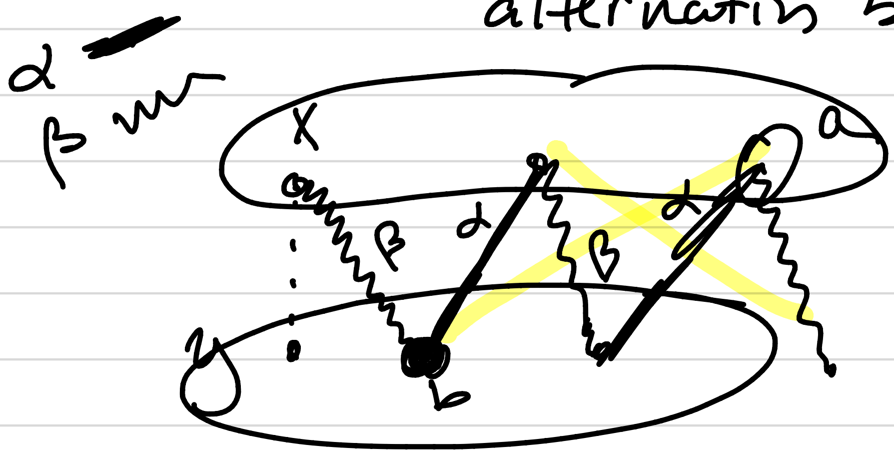
$\alpha :=$ a missing color at x

$\beta :=$ missing color at y

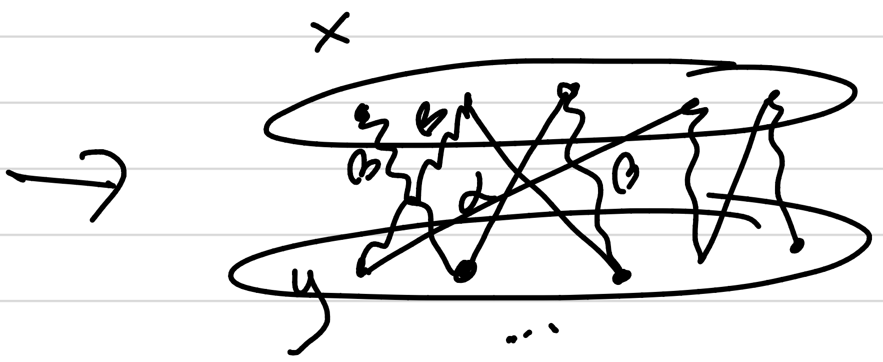
$\in [\Delta] - \{c(xz) : xz \in E(G-e)\}$

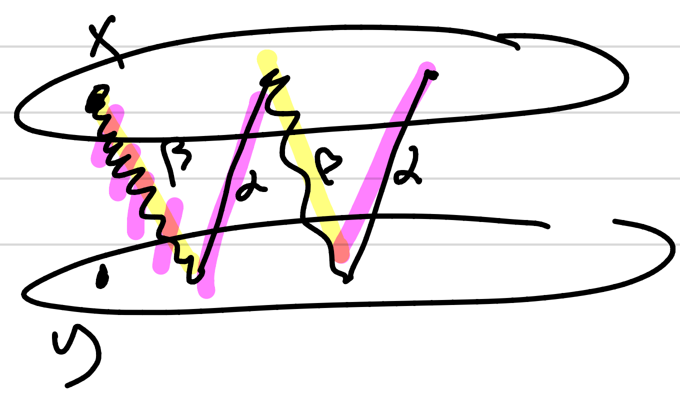
If $\alpha = \beta$, $c(e) = \alpha = \beta$ ✓

Otherwise: Pick a walk starting at x using edges alternating between color β & color α



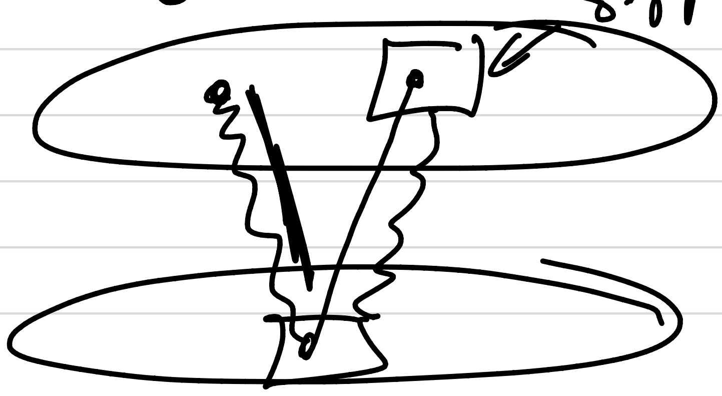
- This is a path
- That does not contain y.





- Now, recolor all edges on path (exchange α & β)

Now, w/ new coloring x and y both are missing color β.
So assign $c(e) = \beta$. stopped



Thm 5.3.2 (Vizing)

$$\Delta(G) \le \chi'(G) \le \Delta(G)+1$$

Pf.

$c : \frac{V}{E} \to [k]$

$\sigma \in \mathrm{Sym}[k]$

$\sigma \circ c$ is also a k-coloring of $\frac{V}{E}$