Theorem (Brooks, 1941) Let G be a connected graph that is not complete and is not an odd cycle. Then $\chi(G) \leq \Delta(G)$.

Proof: The only connected graphs with $\Delta \in \{0,1\}$ are complete. Observe that if $\Delta 2$, then G is a path or a cycle. Since G is not an odd cycle, $\Delta \leq 2$ implies G is bipartite and $\chi(G) \leq 2 = \Delta(G)$. So, we suppose that $\Delta(G) = 3$.

Case 1: There exists a vertex $v \in V$ such that $d(v) \leq \Delta$.

Let T be a spanning tree of G constructed as a breadth first search starting with v as its root. Order the vertices of G in the reverse order in which they are added to the tree. So $v = v_n$ and $N(v)$ will be $v_{n-1}, v_{n-2}, \cdots, v_{n-k}$ where $k = d(v)$. Apply a greedy coloring algorithm to the vertices in this order. Observe that for every $1 \leq i \leq n-1$, the vertex v_i is guaranteed to have a neighbor v_j such that $j > i$. So, at most $\Delta - 1$ neighbors of v_i have been assigned colors when v_i is reached by the coloring algorithm. Thus, a color from the set $\{1, 2, \cdots, \Delta\}$ is available. Since $d(v_n) < \Delta$, there is always a color available for v_n.

Case 2: For every vertex $v \in V$, $d(v)\Delta$. (i.e. G is regular).

Subcase 2.1 $\kappa(G) = 1$

Let v be a cut-vertex of G and let C be a component of $G - v$. Let $H = G[C \cup \{v\}]$. Observe that $d_H(v) < \Delta$. Thus, using the technique from Case 1, we can color H with at most Δ colors and without loss of generality, we can assume v is colored with color Δ. Since C was chosen arbitrarily, we can color every component of $G - v$ and v with Δ colors such that v is colored with color Δ. So the colorings of all of the components separately form a coloring of all of G.

Subcase 2.2 G is 2-connected.

Claim: There exist a triple of vertices x, y, z such that $x, z \in N(y)$ and $G - \{x, z\}$ is connected.

If the Claim is true, we can order the vertices by $v_1 = x, v_2 = z, v_n = y$ and the remaining vertices are ordered according to a spanning tree of $G - \{x, z\}$ constructed as a breadth first search form y. By forcing x and y to have the same color, we guarantee that when a greedy coloring algorithm reaches v_n on the the Δ colors will be available. All vertices v_i for $3 \leq i \leq n-1$ are guaranteed to have at least one neighbor, v_j, such that $i < j$, so these, too, can be assigned one of Δ colors.

Proof of Claim: If G contains a vertex x such that $G - x$ is 2-connected, then any path xyz to a vertex $z \notin N(x)$ satisfies the claim. So we suppose that for every vertex y, $G - y$ is 1-connected. Thus, $G - y$ has a block and cut-vertex structure. Since G is 2-connected, y must be adjacent to a non-cut-vertex in every end-block in the block graph of $G - y$. Pick x and z to be neighbors of y in two different end-blocks that are not cut-vertices. Since x and z are not cut-vertices, $G - \{x, y, z\}$ is still connected. Since $\Delta \geq 3$, y must have at least one edge to $G - \{x, z\}$ and the claim follows.