1. Prove that for every graph $G$, there exists an order of the vertex set of $G$ such that a greedy algorithm using this ordering will use $\chi(G)$ colors.

2. For every $n \geq 3$, construct a bipartite graph on $2n$ vertices and an ordering of the vertex set such that the greedy algorithm will use $n$ colors (as opposed to the optimal 2 colors). Give a justification.

3. A $k$-chromatic graph $G$ is called **critical** if $\chi(G-v) < k$ for every vertex $v \in G$.

   (a) Characterize critical 2-chromatic graphs.

   (b) Find an example of a critical 3-chromatic graph.

   (c) Prove that for $k \geq 3$ every critical $k$-chromatic graph is $(k-1)$-edge-connected.

   (d) Characterize the set of critical 3-chromatic graphs.

4. The **clique number** of a graph, denoted by $\omega(G)$, is the largest $r$ such that $K^r \subseteq G$. The **independence number** of a graph, denoted by $\alpha(G)$, is the largest $r$ such that $G$ contains an independent set of vertices of cardinality $r$.

   (a) Determine $\omega(G)$ and $\alpha(G)$ for the graphs below. Answers are sufficient. No justification required.

      i. $P^m$ for $m \geq 1$

      ii. $C^k$

      iii. $K_{m,n}$ where $m \leq n$

      iv. $K^n$

   (b) Prove that $\chi(G) \geq \max\{\omega(G), |G|/\alpha(G)\}$.

5. Prove or Disprove: Every $k$-chromatic graph $G$ has a $k$-coloring in which some color class has at least $\alpha(G)$ vertices.

6. Assume that $H$ is a $k$-chromatic triangle-free graph and the $G$ is obtained from $H$ by Mycielski's Construction.

   (a) Prove that $G$ is also triangle-free.

   (b) Prove that $G$ is $(k+1)$-colorable.

7. Describe the topic of your project and what source(s) you have found.