MATH 663
Fall 2017
Midterm

Solutions

Books and notes are not allowed. There are 9 problems worth a total of 100 points. You have two hours to complete the exam.

1. (10 points) Determine whether or not the sequence $5, 5, 4, 4, 3, 3, 2, 1, 1$ is graphic. If the sequence is graphic, demonstrate a graph with this degree sequence. If the sequence is not graphic, give a well-defended argument that it is not. Reference any theorems you are using.

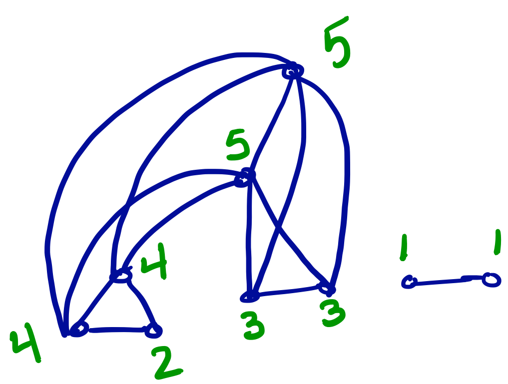
$5\ 5\ 4\ 4\ 3\ 3\ 2\ 1\ 1$

$4\ 3\ 3\ 2\ 2\ 2\ 1\ 1$

$2\ 2\ 1\ 1\ 2\ 1\ 1 = 2\ 2\ 2\ 1\ 1\ 1\ 1$ ← This is graphic.

So This is graphic.

Answer: The sequence is graphic



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−2 : Going through H-H and a typo resulting in wrong answer
−5 : asserting seq. graphical w/o any explanation or a graph demonstration.

2. (10 points) In the matrix below, $a_{i,j}$ gives the weight of the edge between x_i to y_j for a complete bipartite graph $K_{5,5}$. Use the Hungarian Algorithm to find a maximum weight matching (or a transversal of maximum sum) in this graph. Prove that your answer is correct by exhibiting a solution to the dual problem.

$$\begin{array}{cc|c}
 & & \begin{matrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \\
\begin{matrix} 5 \\ 6 \\ 4 \\ 6 \\ 4 \end{matrix} & \begin{matrix} 6 \\ 7 \\ 5 \\ 7 \\ 5 \end{matrix} & \begin{pmatrix} 3 & 5 & 5 & 3 & \boxed{6} \\ 5 & \boxed{6} & 4 & 2 & 7 \\ 1 & 3 & 4 & \boxed{4} & 5 \\ \boxed{7} & 6 & 5 & 5 & 7 \\ 5 & 3 & \boxed{5} & 4 & 1 \end{pmatrix}
\end{array}$$

excess matrix

$$\begin{matrix} 3 & 1 & 1 & 3 & 0 \\ 2 & 1 & 3 & 5 & 0 \\ 4 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 0 & 1 & 4 \end{matrix}$$

min excess = 1

excess matrix

$$\begin{array}{c|ccccc}
x_1 & 3 & 0 & 1 & 2 & \boxed{0} \\
x_2 & 2 & \boxed{0} & 3 & 4 & 0 \\
x_3 & 4 & 1 & 1 & \boxed{0} & 0 \\
x_4 & \boxed{0} & 0 & 2 & 1 & 0 \\
x_5 & 0 & 1 & \boxed{0} & 0 & 4 \\
\hline
 & y_1 & y_2 & y_3 & y_4 & y_5
\end{array}$$

matching w/ max weight

edge	weight
$x_5 y_3$	5
$x_4 y_1$	7
$x_3 y_4$	4
$x_2 y_2$	6
$x_1 y_5$	6
total weight	28

cover cost (in red)

$5+6+4+6+4+1+1+1=28$

Since the cover cost equals the matching weight, matching weight is a maximum

-4 max matching but no wgted cover

-8 one step through algorithm

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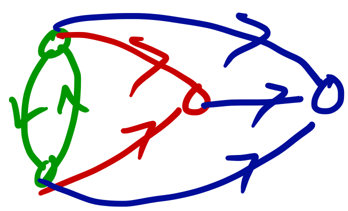
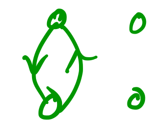
3. (a) (2 points) Define *strong component* in a digraph.

A strong component is a maximal strongly connected subgraph.

(b) (3 points) Give an example of a directed graph on 4 vertices with exactly 3 strong components.



or



(c) (2 points) Define a *tournament*.

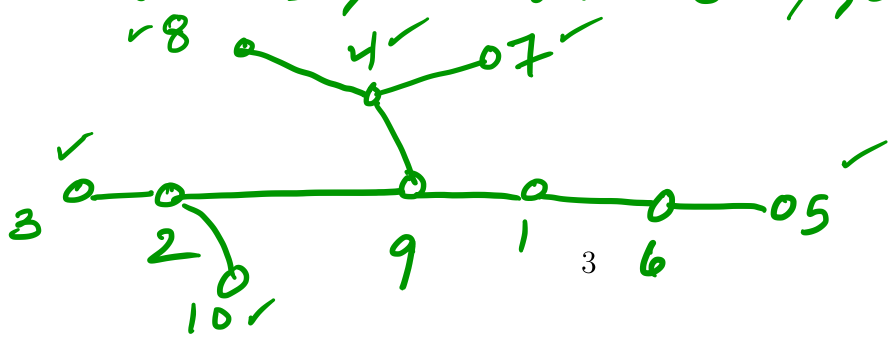
A tounament is an orientation of K_n.

(d) (3 points) Explain why it is not possible to construct a tournament on 4 vertices with exactly 3 strong components.

In a tounament, every nontrivial strong component must contain at least 3 vertices. Since the components cannot all be trivial, one must have 3. But this leaves one vertex for two components.

4. (10 points) Find a tree with Prüfer Code 2,6,1,9,4,4,9,2.

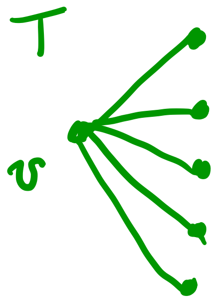
$n=10$ vertices w/ end vertices: 3, 5, 7, 8, 10, 6, 1, 4



5. (a) (2 points) State the definition of a *tree*.

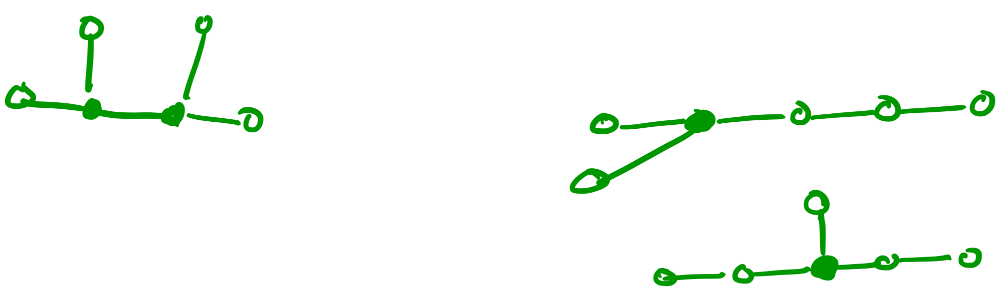
An acyclic connected graph.

(b) (4 points) Give an example of a tree T on 6 vertices and a set $S \subseteq V(T)$, such that S is a maximal independent set of vertices that is *not* a maximum independent set of vertices.



$S = \{v\}$

(c) (4 points) List all nonisomorphic trees on 6 vertices with maximum degree 3. Do not list any isomorphism class more than once.



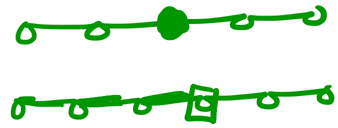
6. (10 points) Prove that every simple graph on at least two vertices has at least two vertices of the same degree.

If G has n vertices and G is simple, then the set of possible values for $d(v)$ is $\{0, 1, \dots, n-1\}$.

But we see that the values $n-1$ and 0 cannot be used in the same graph G. Thus, there are at most $n-1$ available values for $d(v)$. Since G has n vertices, at least one value must be repeated.

7. (16 points) For $K_{m,n}$ and P_n, find

thinking



(a) the radius of the graph.

$\text{rad}(K_{m,n}) = 2$

(If $m=1$ or $n=1$, then $\text{rad}(K_{m,n}) = 1$)

$\text{rad}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$

(b) the diameter of the graph

$\text{diam}(K_{m,n}) = 2$

(unless $m=n=1$, in which case $\text{diam}(K_{1,1}) = 1$.)

$\text{diam}(P_n) = n-1$

(c) the center of the graph

center of $K_{m,n}$ is $K_{m,n}$ (If $m=1$ or $n=1$, then the center of $K_{m,n}$ is one vertex.)

center of P_n is K_1 if n odd
K_2 if n even

(d) the number of edges in a maximum matching.

$\alpha'(K_{m,n}) = \min\{m,n\}$

$\alpha'(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$

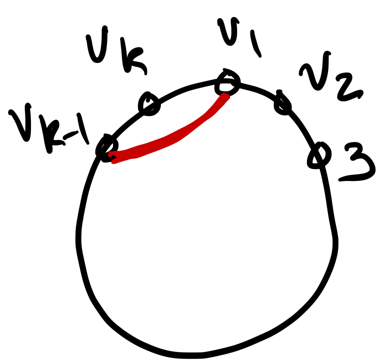
8. (12 points) Assign integer weights to the edges of K_n. Prove that the total weight on every cycle is even if and only if the total weight on every triangle is even. (Note: Make the logical structure of your argument clear.)

$\Rightarrow$: If the weight of every cycle is even, then the weight of every 3-cycle is even.

+4

$\Leftarrow$: Assume the weight of every 3-cycle is even. Proceed by induction on the number of vertices in the cycle. That is, assume all cycles on fewer than k vertices have even weight. Let C_k be a cycle with vertex set: $v_1 v_2 v_3 \ldots v_k v_1$. (See picture below) Since $k > 3$, $v_{k-1} \not\leftrightarrow v_1$ in C_k.

+8



Consider $C_k + v_1 v_{k-1}$. By the inductive hypothesis, the cycles $C_3 = v_1 v_{k-1} v_k$ and $C_{k-1} = v_1 v_2 v_3 \ldots v_{k-1} v_k$ both must have even weight.

Thus,

$$w(C_k) = w(C_3) + w(C_{k-1}) - 2w(e)$$

is even because the three numbers in the right-hand sum are even.

9. (a) (2 points) Define $\alpha(G)$, the independence number of the graph G.

$\alpha(G)$ is the maximum cardinality of a set of independent vertices of G.

(b) (2 points) Explain what the symbol $\Delta(G)$ means.

ΔG denotes the maximum degree of G.

named S

(c) (8 points) Prove that for every graph G, $\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$.

We will construct an independent set of vertices of cardinality $\frac{n}{\Delta+1}$ which will suffice to prove the statement. Pick an arbitrary vertex $v \in G$. Add v to S and delete v and $N(v)$ from G. Note, at most $\Delta+1$ vertices were deleted. Repeat on $G-v-N(v)$. We know S is independant since the neighbors of v were excluded from S when v is added. This algorithm must run at least $\frac{n}{\Delta+1}$ times. Thus $|S| \geq n/\Delta+1$.