MATH 663
FALL 2017
FINAL EXAM

NAME:

Books and notes are not allowed. There are 8 problems worth a total of 100 points. You have two hours to complete the exam. NOTE: The space provided for an answer should not be taken as an indication of the amount of writing required to answer the question.

1. (10 points)

   (a) Use Havel-Hakimi Theorem to determine if the sequence below is graphic:

   $$8, 8, 4, 4, 4, 4, 4, 4, 4, 2, 2$$

   (b) Does there exist an r such that the following sequence is the degree sequence of a **tree**? If so, show your answer is correct. If not, explain why.

   $$r, 4, 4, 3, 3, 1, 1, 1, 1, 1$$

   Just for clarity: there are three 3's, there are six 1's, and one other nonnegative integer r. While r is listed at the front, we do *not* assume $r \geq 3$.

2. (4 points each) Give an example of each of the following or give a brief explanation for why one does not exist.

   (a) a simple graph G such that $\kappa(G) + 2 = \kappa'(G)$

   (b) a simple graph G and set $S \subseteq V(G)$ such that $|S| \geq 5$ and $|S| < o(G - s)$.

   (c) a 3-regular graph on 10 vertices

   (d) a directed graph on 6 vertices with exactly two strong components such that its underlying graph is connected

   (e) a nonplanar graph on 7 vertices.

3. (a) (5 points) Carefully state Euler's Formula.

(b) (10 points) Let G be an n-vertex, simple, planar graph with girth k. Prove that G has at most $(n-2)\dfrac{k}{k-2}$ edges.

4. (15 points)

(a) Define $\chi(G)$.

(b) Define $\alpha(G)$.

(c) Prove that for every graph G, $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

(d) Prove or disprove the following statement:

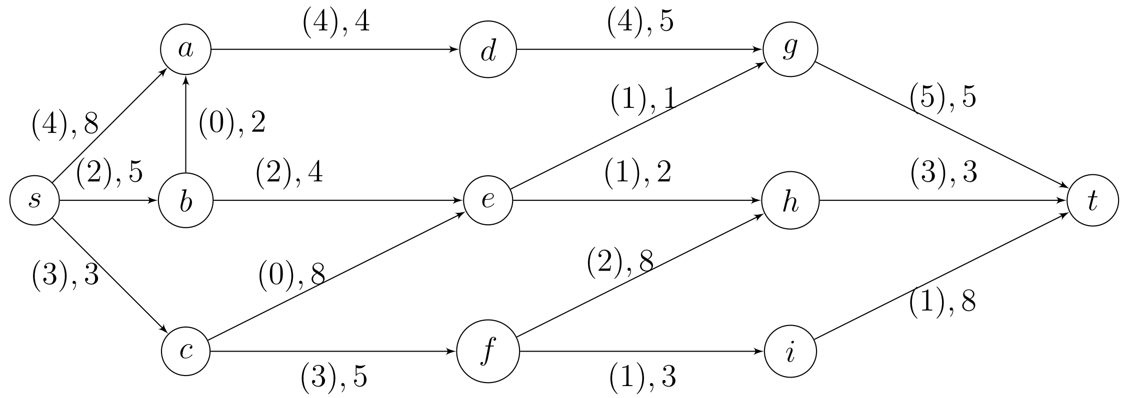
> Every k-chromatic graph G has a proper k-coloring in which some color class has $\alpha(G)$ vertices.

5. (15 points)

   (a) State Menger's Theorem

   (b) Use Menger's Theorem to prove that if G is 2-connected, then every pair of edges in G lies on a common cycle.

6. (10 points) Given the network below, complete a TWO iterations of the Ford-Fulkerson Algorithm as described in our book. State clearly the output of the Algorithm. Note that the number in parentheses represents the flow on the edge and the second number represents the capacity of the edge.



7. (10 points) In the matrix below, $a_{i,j}$ gives the weight of the edge between x_i to y_j for a complete bipartite graph $K_{5,5}$. Use the Hungarian Algorithm to find a maximum weight matching (or a transversal of maximum sum) in this graph. Prove that your answer is correct by exhibiting a solution to the dual problem.

$$\begin{pmatrix} 12 & 9 & 9 & 7 & 7 \\ 10 & 8 & 8 & 7 & 7 \\ 10 & 7 & 7 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 \end{pmatrix}$$

8. (5 points) Prove that every simple graph on at least two vertices has two vertices of equal degree.