MATH 663
Spring 2010
In-Class Midterm
Selected Solutions

3.b. Prove that the strong components of a digraph are pairwise disjoint.

Let C_1 and C_2 be strong components of a digraph. Let $v \in V(C_1) \cap V(C_2)$. Then for every $x \in V(C_1)$ and $y \in V(C_2)$, there exists an x, v-path in D and a v, y-path in D. Thus, there exists an xy-path in D. By a symmetric argument, D must contain a y, x-path. But the maximality of C_1 now implies y is in C_1. Similarly x is in C_2. Thus, $C_1 = C_2$ and strong components must be pairwise disjoint.

4.b. How many trees on n vertices (with vertex labels $1, 2, \cdots, n$) have vertex n as a leaf?

The Prüfer code for such a tree will be a string such that n does NOT appear. The number of strings of length $n-2$ from an alphabet of $n-1$ letters is: $(n-1)^{n-2}$.

5.b. Give an example of a simple graph G on n vertices for $n \geq 3$ such that diam(G)=rad(G)=2.

A biclique will work.

6.b. Use the König-Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. (Recall $e(G)$ is the number of edges and $\Delta(G)$ is the maximum degree of G.)

The Q be a minimal edge cover of the bipartite graph G. Since each vertex of Q can cover at most $\Delta(G)$ edges, $|Q| \geq e(G)/\Delta(G)$. From the König-Egerváry Theorem we know that any maximal matching M must have the same number of edges has Q has vertices. So, $M \geq e(G)/\Delta(G)$.

7.b. Let G be a simple graph such that $\delta(G) \geq 3$. Prove that G contains an even cycle.

Let G be a simple graph such that $\delta(G) \geq 3$. Let $P = u_1, u_2, \cdots, u_n$ be a maximal path in G. Since P is maximal and $d(u_1) \geq 3$, u_1 must have two additional neighbors on P other than u_2. If either of these neighbors have even index, say u_{2k}, then $x_1, x_2, \cdots, x_{2k}, x_1$ forms an even cycle. If u_1 has two neighbors with odd index, say u_{2k+1} and u_{2j+1}, then $x_1, u_{2k+1}, u_{2k+2}, \cdots, u_{2j+1}, x_1$ forms an even cycle on $2j + 1 - (2k - 1) = 2(j - k)$ vertices.