Comprehensive Exam: Applied Combinatorics and Graph Theory
Spring 2010

Books and notes are not allowed. There are 8 problems all of equal weight. Complete as many as possible. You have two hours to complete the exam.

1. Let G be a simple graph with $\delta(G) = k \geq 2$. Let P be a maximal path in G. Prove that if $n(P) \leq 2k$, then the induced subgraph $G[V(P)]$ is Hamiltonian.

2. Prove that a graph G is connected if and only if for every partition of its vertices into two nonempty sets, there is an edge with endpoints in both sets.

3. (a) State (carefully) Euler's Formula.

   (b) Use Euler's Formula to show that every simple, triangle-free, planar graph has no more than $2n - 4$ edges, where n is the number of vertices in the graph.

   (c) Give an example of a simple, triangle-free, planar graph on n vertices (for arbitrary $n \geq 3$) that achieves this upper bound. (That is, show your result in part b is sharp.)

4. (a) State Brooks' Theorem.

   (b) Give an example to show the result is sharp for all n.

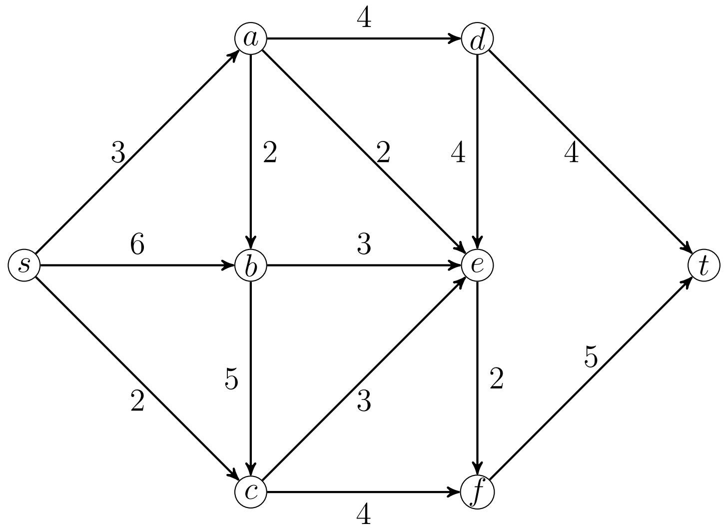
   (c) State Vizing's Theorem.

   (d) Give an example to show the result is sharp for all n.

5. Recall that in a digraph, a **king** is a vertex from which every vertex is reachable by a path of length at most 2. Prove that every tournament has a king.

6. Let T be a minimum weight spanning tree in a weighted connected graph G. Prove that T omits some heaviest edge from every cycle in G.

7. In the network below, find a maximum flow from s to t and prove that your answer is correct.



8. Let G be a weighted graph isomorphic to $K_{5,5}$ with bipartition X, Y. Let the matrix M below be the matrix of edge weights of G. Find a maximum weighted matching in G by exhibiting a minimum cost cover.

matrix M

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\ \begin{pmatrix} 3 & 5 & 4 & 3 & 3 \\ 2 & 2 & 5 & 4 & 3 \\ 6 & 5 & 2 & 4 & 1 \\ 4 & 3 & 5 & 3 & 4 \\ 6 & 2 & 7 & 5 & 3 \end{pmatrix} \end{array}$$