Directions:

• You have 2 hours to complete all 6 of the problems below. • Books, notes or other aids are not allowed. • To receive full credit, proofs must be formal. • Paper will be provided for you. Please put your answer to each problems on a separate sheet of paper.

1. Prove that every automorphism of a tree fixes a vertex or an edge.

2. Let G be a simple planar graph on n vertices with girth k. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges.

3. Let $G = (A \cup B, E)$ be a bipartite graph with partite sets A and B such that $|A| = |B|$ and $E \neq \emptyset$. Prove that if $|N(X)| > |X|$ for every nonempty $X \subseteq A$, then every edge of G lies on a 1-factor.

4. Let G be a graph on n vertices.

   (a) Prove that if $\delta(G) \geq 3$, then G contains a cycle with a chord. Recall that a **chord** in a cycle is an edge between two vertices on the cycle that is not a cycle edge.

   (b) Prove that if $n \geq 4$ and $|E(G)| \geq 2n-3$, then G contains a chord.

5. Prove that in e very 2-coloring of the edges of K_n (for $n \geq 3$), ther is either a monochromatic hamiltonian cycle or a hamiltonian cycle with exactly two monochromatic arcs. (By "exactly two monochromatic arcs" we mean that the hamiltonian cycle can be labelled $C = v_1 v_2 \cdots v_n v_1$ such that all the edges on the path $v_1 v_2 \cdots v_i$ are one color and the remaining edges $v_i v_{i+1} \cdots v_n v_1$ are the same color.)

6. Let $S_1, S_2, \cdots, S_m$ be a collection of finite sets such that $2 \leq |S_1| \leq |S_2| \leq \cdots \leq |S_m|$. Define a graph G with vertex set $V = S_1 \times S_2 \times \cdots \times S_m$ such that m-tuples u and v are adjacent if and only of they differ in every coordinate. Determine $\chi(G)$.