1. Prove that a graph is bipartite if and only if every **induced** cycle has even length.

   Proof:

   ($\Longrightarrow$:) Suppose that G is bipartite. Thus, all cycles in G are even. Thus, all induced cycles are even.

   ($\Longleftarrow$:) Suppose that every induced cycle in G is even. It is sufficient to show that G contains no odd cycle. Proceed by contradiction and suppose that G contains an odd cycle. Among all odd cycles in G, choose C to be a smallest odd cycle. By assumption, C cannot be induced. Thus, C must have a chord. But any chord in a cycle of odd length can be used to form two smaller cycles on $V(C)$, one of even length and one of odd length. This contradicts the choice of C as a cycle of smallest possible odd order. Thus, G can contain no odd cycles. Thus, G is bipartite.

2. (a) State Tutte's Theorem.

   A graph $G = (V, E)$ contains a 1-factor if and only if for every $S \subseteq V$, $q(G-S) \leq |S|$, where $q(G-S)$ is the number of components of $G-S$ of odd cardinality.

   (b) Use Tutte's Theorem to prove that every 3-regular graph with no bridges must have a 1-factor.

   Proof: Suppose G is a cubic, bridgeless graph. We will show that G satisfies Tutte's condition.

   Let $S \subseteq V$ and let C be a component of $G-S$ of odd cardinality. Observe that $\sum_{v \in V(C)} d_G(v) = 3 \cdot |C|$, which must be odd since 3 and $|C|$ are both odd. But $\sum_{v \in V(C)} d_C(v)$ is even because the degree sum of the vertices in any graph is even. Thus, the number of edges from C to S is odd. Since G is bridgeless, the number of edges must be at least 3.

   Thus, the number of edges from odd components of $G-S$ to S is at least $3 \cdot q(G-S)$. Since G is cubic, the number of edges between all components of $G-S$ and S can be at most $3 \cdot |S|$. Thus, $3 \cdot |S| \geq 3 \cdot q(G-S)$, and Tutte's condition applies.

3. Let G be a graph on n vertices. Recall that $\delta(G)$ denotes the minimum degree of G. Prove that if $\delta(G) \geq (n-1)/2$, then G must be connected.

   Proof: Let G be a graph on n vertices. Let x and y be arbitrary vertices of G. We need to find an xy-path. If x and y are adjacent, the edge xy is the path. If x and y are nonadjacent, then observe that $N(x) \cup N(y) \subseteq G - \{x, y\}$. But,

$$|N(x)| + |N(y)| \geq 2\left(\frac{n-1}{2}\right) = n-1 > n-2 = |G - \{x, y\}|.$$

   So by the Pigeonhole Principle, x and y have a common neighbor which forms xy-path of length 2.

4. Recall that a graph G is critically 2-connected if G is 2-connected by for every $e \in E(G)$, $G - e$ is no longer 2-connected. Prove that every critically 2-connected graph must contain a vertex of degree 2.

   Proof: Suppose that G is critically 2-connected. Since G is 2-connected, we know that G can be constructed by starting with a cycle and iteratively adding paths with disjoint end-vertices. If G is just a cycle, then all vertices are of degree 2. Otherwise, constructing G must require adding a last H-path. Recall from our homework that none of the added paths can be simply an edge because G wouldn't be critically 2-connected. Thus, the last added path has at least three vertices and, therefore, a vertex of degree 2.

5. Suppose G is a connected graph on n vertices. Prove that G has exactly one cycle if and only if G has exactly n edges.

   Proof:

   ($\Longrightarrow$:) Suppose G is a connected graph with exactly one cycle. Let e be any edge on the unique cycle of C. Then $G - e$ must be connected and acyclic. Thus, $G - e$ is a tree on n vertices and thus has $n-1$ edges. Thus, G has n edges.

   ($\Longleftarrow$:) Suppose G is a connected graph with exactly n edges. Since G is connected, it contains a spanning tree, T. So, T has $n-1$ edges. Thus, G has exactly one edge that is not in T. Since, as a tree, T is minimally acyclic, $G = T + e$ would have exactly one cycle.

6. Let $G = (V, E)$ be a 2-connected graph. Prove that for every $a \in V$ there exists some $b \in N(a)$ such that $G - a - b$ is still connected. (To be clear, the graph $G - a - b$ is the graph obtained from G by deleting both the vertex a and its neighboring vertex b.)

   Proof: Let $G = (V, E)$ be a 2-connected graph and $a \in V$, arbitrary. If $G - a$ is 2-connected, then any vertex in $b \in N(a)$ will satisfy the condition that $G - a - b$ is connected.

   If $G - a$ is not 2-connected, then $\kappa(G - a) = 1$. Thus, we know that its block graph is a tree. Let B be one of the end-blocks of the block graph of $G - a$ and let v be the unique cut-vertex in B. Observe that a must have a neighbor in $B - v$ since otherwise v is a cut-vertex of G. Let c be a neighbor of a in $B - v$. Then, $G - a - c$ must be connected since $B - c$ must be connected.