Directions:

- You have 2 hours to complete all 6 of the problems below.
- Books, notes or other aids are not allowed.
- To receive full credit, proofs must be formal.
- Paper will be provided for you. Please put your answer to each problems on a separate sheet of paper.

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1. Prove that a graph is bipartite if and only if every **induced** cycle has even length.
2. (a) State Tutte's Theorem.
   (b) Use Tutte's Theorem to prove that every 3-regular graph with no bridges must have a 1-factor.
3. Let G be a graph on n vertices. Recall that $\delta(G)$ denotes the minimum degree of G. Prove that if $\delta(G) \geq (n-1)/2$, then G must be connected.
4. Recall that a graph G is critically 2-connected if G is 2-connected by for every $e \in E(G)$, $G-e$ is no longer 2-connected. Prove that every critically 2-connected graph must contain a vertex of degree 2.
5. Suppose G is a connected graph on n vertices. Prove that G has exactly one cycle if and only if G has exactly n edges.
6. Let $G = (V,E)$ be a 2-connected graph. Prove that for every $a \in V$ there exists some $b \in N(a)$ such that $G-a-b$ is still connected. (To be clear, the graph $G-a-b$ is the graph obtained from G by deleting both the vertex a and its neighboring vertex b.)