1. Let G be a simple graph such that $\chi(G) = k$. Let $c : V(G) \to [k]$ be a k-coloring of the vertex set of G. Prove that for every $i \in [k]$, there exists at least one vertex assigned color i that is adjacent to at least one vertex of each of the other $k-1$ colors.

   Proof: Let G be a simple graph with chromatic number k and k-coloring c. For every $i \in [k]$, let V_i be the vertices in color class i.

   Observe that if $c(x) = i$ and x has no adjacencies in V_j, for $i \neq j$, then $V_j \cup x$ is an independent set and so x can be reassigned to color j.

   If there exists a color class V_i such that for every $x \in V_i$ there exists a color $j \neq i$ such that x has no adjacencies in V_j, then every vertex in V_i can be assigned to some new color. This would imply that $\chi(G) < k$, a contradiction.

2. Prove that if G is r-regular and $\kappa(G) = 1$, then $\chi'(G) > r$. (Recall that $\kappa(G)$ is the connectivity of a graph.)

   Proof: Suppose G is r-regular and $\kappa(G) = 1$. Since $\kappa(G) = 1$, it follows that $r \geq 2$.

   Let v be a cut vertex of G with neighbors x and y in distinct components of $G - v$. Without loss of generality suppose that vx is assigned color 1and vy is assigned color 2.

   If $\chi'(G) = r$, then the set of edges assigned colors 1 and 2 form a 2-regular graph. Thus, v lies on a cycle which contracts the fact that v is a cut vertex.

3. (a) Describe the graph, $T^r(n)$, the Turán graph, as described in our text.

   It's a complete r-partite graph on n vertices such that the vertex classes are as even as possible.

   (b) State Turán's Theorem.

   The maximum number of edges in a graph G on n vertices that does not contain K^{r+1} as a subgraph is $|E(T^r(n))|$, and this extremal graph is unique.

   (c) Show that $|E(T^r(n))| = \binom{r}{2} + (n-r)(r-1) + |E(T^r(n-r))|$.

   Observe that $K^r \subseteq T^r(n)$. Call this subgraph H and call $T^r(n) - V(H) = G$. Thus, the number of edges in H is $\binom{r}{2.}$ Each of the $n-r$ vertices in G is adjacent to exactly $r-1$ of the vertices of H. Thus, there are $(n-r)(r-1)$ edges between $V(H)$ and $V(G)$. Finally we observe that $G = T^r(n-r)$.

4. Suppose that G is planar and triangle-free. Prove that G has a vertex of degree at most 3.

   Proof: Note that it is sufficient to prove that no connected component of G has a vertex of degree at most 3 since if the statement applies to an arbitrary component of G, it applies to all of G. Thus, let C be a particular connected plane component of the plane graph G. Let n, e and f represent the number of vertices, edges and faces of the plane graph C. Since G is triangle-free, C is triangle-free. Thus, every face of C has at least four edges on its frontier which implies that $2e \geq 4f$ or $f \leq \frac{1}{2}e$.

   Suppose for a contradiction that $\delta(G) \geq 4$. Then, $2e \geq 4n$, or $n \leq \frac{1}{2}e$.

   Applying Euler's Formula to C we obtain:

   $$2 = n - e + f \leq \frac{1}{2}e - e + \frac{1}{2}e = 0,$$

a contradiction.

5. Suppose that G is a graph such that every subgraph of G contains a vertex of degree k or less. Prove that the chromatic number of G is at most $k+1$.

   Proof Suppose that G is a graph such that every subgraph of G contains a vertex of degree k or less.

   Let v_1 be a vertex of $G = G_1$ of degree at most k. Let v_2 be a vertex of $G_2 = G_1 - v_1$ of degree at most k in G_2. Proceed iteratively so that v_i is a vertex in $G_i = G_{i-1} - \{v_1, v_2, \cdots, v_{i-1}\}$ of degree at most k.

   Now apply a greedy algorithm to $V(G)$ in the order $v_n, v_{n-1}, \cdots v_2, v_1$.