Directions:

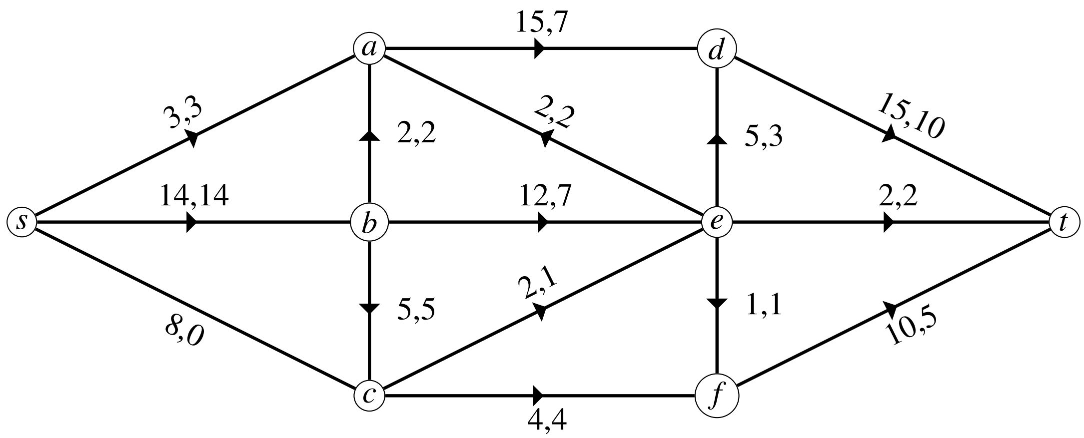
- You have 2 hours to complete all 6 of the problems below.
- Books, notes or other aids are not allowed.
- To receive full credit, proofs must be formal.
- Paper will be provided for you. Please put your answer to each problems on a separate sheet of paper.

---

1. Let G be a simple graph such that $\chi(G) = k$. Let $c : V(G) \to [k]$ be a k-coloring of the vertex set of G. Prove that for every $i \in [k]$, there exists at least one vertex assigned color i that is adjacent to at least one vertex of each of the other $k-1$ colors.
2. Prove that if G is r-regular and $\kappa(G) = 1$, then $\chi'(G) > r$. (Recall that $\kappa(G)$ is the connectivity of a graph.)
3. (a) Describe the graph, $T^r(n)$, the Turán graph, as described in our text.
   (b) State Turán's Theorem.
   (c) Suppose that $n \geq 2r$. Show that $|E(T^r(n))| = \binom{r}{2} + (n-r)(r-1) + |E(T^r(n-r))|$.
4. Suppose that G is planar and triangle-free. Prove that G has a vertex of degree at most 3.
5. Suppose that G is a graph such that every subgraph of G contains a vertex of degree k or less. Prove that the chromatic number of G is at most $k+1$.

See the 6th problem on networks on the next page!

6. In the network below, the capacity and flow value for each edge are represented with an ordered pair: (capacity, flow). Assume the capacity of every edge is the same regardless of direction. (So $c(sa) = c(as) = 3$.) Assume the arrows indicate the direction of positive flow. (So, $f(sa) = 3$ and $f(as) = -3$.)



Use the iterative process from the Ford-Fulkerson Theorem to augment the existing flow until a maximum flow is obtained. You can illustrate your iterative process using the figure(s) below. Demonstrate that your flow has maximum value by finding an appropriate vertex cut S.